Stochastic Analysis of the Fractional Brownian Motion

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Abstract

Since the fractional Brownian motion is not a semi–martingale, the usual Ito calculus cannot be used to define a full stochastic calculus. However, in this work, we obtain the Itô formula, the Itô–Clark representation formula and the Girsanov theorem for the functionals of a fractional Brownian motion using the stochastic calculus of variations.

1 Introduction

In engineering applications of probability, stochastic processes are often used to model the *input* of a system. For instance, the financial mathematics requires stochastic models for the time evolution of assets and the queuing networks analysis is based on models of the offered traffic. Hitherto, the stochastic processes used in these fields are often supposed to be Markovian. However, recent studies [8] show that *real inputs* exhibit long-range dependence : the behavior of a real process after a given time t does not only depend on the situation at t but also of the whole history of the process up to time t. Moreover, it turns out that this property is far from being negligible because of the effects it induces on the expected behavior of the global system [12].

Another property that have the processes encountered in applications (at least in communication networks) is the self-similarity (see [8]): their behavior is stochastically the same, up to a space-scaling, whatever the time-scale is – this is to say that the process $\{X_{\alpha t}, t \in [0, 1]\}$ has the same law as the process $\{\alpha^H X_t, t \in [0, 1]\}$, where H is called the Hurst parameter. Several estimations on real data tend to show that H often lies between 0.7 and 0.8 whereas for instance, the usual Brownian motion has a Hurst parameter equal to 0.5 but it is also clear that some real processes have a Hurst parameter less than 0.5 – see [5]. There exist several stochastic processes which are self-similar and exhibiting long-range dependence but the fractional Brownian motion (fBm for short) seems to be one of the simplest.

Definition 1.1. For any H in $(0,1)$, the fractional Brownian motion of index (Hurst parameter) H, $\{W_t^H;\ t \in [0,1]\}$ is the centered Gaussian process whose covariance kernel is given by

$$R_H(s,t) = \mathbf{E}_H\left[W_s^H W_t^H\right] \stackrel{def}{=} \frac{V_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

where

$$V_H \stackrel{def}{=} \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

Since for $H \neq 1/2$, the fBm is not a semimartingale, we can not use the usual stochastic calculus to analyze it, however since it is a Gaussian process, we can apply the stochastic calculus of variations which is valid on general Wiener spaces. Actually, two choices are offered to us : either some well known properties of the standard Brownian motion are used to derive some properties of the fBm or we can proceed by an intrinsic analysis of the fBm. The first approach leads us to the Itô–Clark formula whereas the Itô formula and the Girsanov theorem are more intrinsic results. This paper is organized as follows : in Section 2, we give some results on hypergeometric functions and deterministic fractional calculus which will be useful in the sequel, in Section 3 we give some sample–paths properties of the fractional Brownian motion, in section 4 we introduce the stochastic calculus of variations. It enables us to define several stochastic integrals with respect to the fractional Brownian motion of any order. We can then give the Itô–Clark representation formula and the Girsanov theorem for adapted processes. In the last section, we give Itô formulae for $H > 1/2$ using different stochastic integrals. Throughout the paper, we give two practical applications such as the simulation of sample-paths of the fractional Brownian motion and an estimation problem involving an fBm.

2 Deterministic fractional calculus

The Gauss hypergeometric function $F(a,b,c,z)$ (for details, see [11]) is defined for any a,b, any z, $|z|<1$ and any $c\neq 0,-1,\dots$ by

$$F(a,b,c,z) \stackrel{def}{=} \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \tag{1}$$

where $(a)_0 = 1$ and $(a)_k \stackrel{def}{=} \Gamma(a+k)/\Gamma(a) = a(a+1)\dots(a+k-1)$ is the Pochhammer symbol. If a or b is a negative integer the series terminates after a finite number of terms and $F(a,b,c,z)$ is a polynomial in z. The radius of convergence of this series is 1 and there exists a finite limit when z tends to 1 ($z<1$) provided that $\Re(c-a-b)>0$. For any z such that $|arg(1-z)|<\pi$, any a,b,c such that $\Re(c)>\Re(b)>0$, F can be defined by

$$F(a,b,c,z) \stackrel{def}{=} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}\, du. \tag{2}$$

Given (a,b,c), consider Σ the set of triples (a',b',c') such that $|a-a'|=1$ or $|b-b'|=1$ or $|c-c'|=1$. Any hypergeometric function $F(a',b',c',z)$ with (a',b',c') in Σ is said to be contiguous to $F(a,b,c)$. For any two hypergeometric functions F_1 and F_2 contiguous to $F(a,b,c,z)$, there exists a relation of the type :

$$P_0(z)F(a,b,c,z)+P_1(z)F_1(z)+P_2(z)F_2(z)=0, \text{ for } z,\ |arg(1-z)|<\pi, \tag{3}$$

where for any i, P_i is a polynomial with respect to z. These relations permit to define the analytic continuation of $F(a,b,c,z)$ with respect to its four variables in the domain $\mathbb{C}\times\mathbb{C}\times(\mathbb{C}\backslash\{0,-1,-2,\dots\})\times\{z,|arg(1-z)|<\pi\}$. We will also use other types of relations between different hypergeometric functions, namely :

$$\begin{aligned} F(a,b,c,z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(1-z)^{-a}F(a,c-b,1+a-b,1/(1-z)) \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(1-z)^{-b}F(b,c-a,1-a+b,1/(1-z)), \end{aligned} \tag{4}$$

for any z such that $|arg(1-z)|<\pi$ and $a-b\neq 0,\pm1,\pm2,\dots$. We now consider some basic aspects of the deterministic fractional calculus – the main reference for this subject is [13].

Definition 2.1. Let $f \in L^1([a,b])$, the integrals

$$(I^\alpha_{a^+} f)(x) \stackrel{def}{=} \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt \ , \ x \geq a,$$
$$(I^\alpha_{b^-} f)(x) \stackrel{def}{=} \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(x-t)^{\alpha-1} dt \ , \ x \leq b,$$

where $\alpha > 0$, are respectively called right and left fractional integral of the order α.

For any $\alpha \geq 0$, any $f \in L^p([0,1])$ and $g \in L^q([0,1])$ where $p^{-1}+q^{-1} \leq \alpha$, we have :

$$\int_0^t f(s)(I^\alpha_{0^+} g)(s) \ ds = \int_0^t (I^\alpha_{t^-} f)(s) g(s) \ ds. \tag{5}$$

Definition 2.2. For f given in the interval $[a,b]$, each of the expressions

$$(\mathcal{D}^\alpha_{a^+} f)(x) \stackrel{def}{=} \left(\frac{d}{dx}\right)^{[\alpha]+1} I^{1-\{\alpha\}}_{a^+} f(x),$$
$$(\mathcal{D}^\alpha_{b^-} f)(x) \stackrel{def}{=} \left(-\frac{d}{dx}\right)^{[\alpha]+1} I^{1-\{\alpha\}}_{b^-} f(x),$$

are respectively called the right and left fractional derivative (proved they exist), where $[\alpha]$ denotes the integer part of α and $\{\alpha\} = \alpha - [\alpha]$.

A sufficient condition for f to be α-differentiable almost everywhere (with respect to the Lebesgue measure on $[a,b]$) is that f is continuously differentiable of any integer order less than $[\alpha]$ and that $f^{([\alpha])}$ is absolutely continuous. Note that $(\mathcal{D}^1_{a^+} f)$ coincides with the usual derivative of absolutely continuous function. Moreover, if f is α-differentiable then f is β-differentiable for any $\beta \leq \alpha$.

Proposition 2.1. *For $\alpha \in \mathbb{C}$ such that $\Re(\alpha) > 0$, we have :*

$$\mathcal{D}^\alpha_{a^+} I^\alpha_{a^+} f = f \ \textit{for} \ f \in L^1([a,b]),$$
$$I^\alpha_{a^+} \mathcal{D}^\alpha_{a^+} f = f \ \textit{for} \ f \in I^\alpha_{a^+}(L^1([a,b])).$$

As a consequence, we will often denote $\mathcal{D}^\alpha_{a^+}$ by $I^{-\alpha}_{a^+}$. Moreover, for $p \geq 1$, the latter proposition also induces that a function f in $I^\alpha_{a^+}(L^p([a,b]))$ is α–differentiable (for a reciprocal of this assertion, see [13, page 232]) and hence continuous. Some extra work proves that such a function is Hölder continuous of order $\alpha - 1/p$ – [13, Thm 3.6, page 67]. The next theorem will be a key result for the sequel :

Theorem 2.1 (cf [13, page 187]).
For $H \in (0,1)$, consider the integral transform :

$$(K_H f)(t) =$$
$$\Gamma(H+1/2)^{-1} \int_0^t (t-x)^{H-1/2} F(H-1/2, 1/2-H, H+1/2, 1-t/x) f(x) dx. \tag{6}$$

K_H is an isomorphism from $L^2([0,1])$ onto $I_{0+}^{H+1/2}(L^2([0,1]))$ and

$$K_H f = I_{0+}^{2H} x^{1/2-H} I_{0+}^{1/2-H} x^{H-1/2} f \quad \text{for } H \le 1/2,$$
$$K_H f = I_{0+}^{1} x^{H-1/2} I_{0+}^{H-1/2} x^{1/2-H} f \quad \text{for } H \ge 1/2.$$

Note that if $H \ge 1/2$, $r \to K_H(t,r)$ is continuous on $(0,t]$ so that we can include t in the indicator function.

3 Properties of the Fractional Brownian Motion

Using the Kolmogorov criterion, it is easy to see that for any H, there exists a version of W^H whose sample-paths are continuous and with standard techniques, it can also be shown that sample-paths are nowhere differentiable (see [10]). Furthermore, the form of the covariance kernel entails that W^H has stationary increments and that the process is self-similar in the sense that

$$\{W^H_{\alpha t},\ t \in [0,1]\} \stackrel{d}{=} \{\alpha^H W^H_t,\ t \ge 0\}.$$

Note that increments are independent only when $H = 1/2$, for $H > 1/2$, increments are positively correlated and for $H < 1/2$ they are negatively correlated. This difference of behavior between the cases $H < 1/2$ and $H > 1/2$ can also be seen in the regularity of sample-paths as show the next figure and the next theorem.

Theorem 3.1. *Let $H \in (0,1)$, the sample-paths of W^H are a.s. Hölder continuous only of order less than H.*

Proof. Since, for any $\alpha \ge 0$, we have

$$\mathbf{E}_H \left[|W^H_t - W^H_s|^\alpha \right] = C_\alpha |t-s|^{H\alpha},$$

the Kolmogorov criterion implies that the sample-paths of W^H are almost surely Hölder continuous of any order less than H.

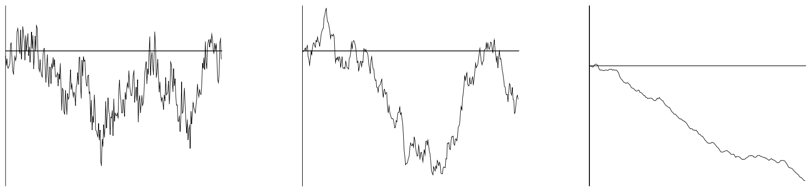

Figure 1: Typical sample-path for $H = 0.2,\ H = 0.5,\ H = 0.8$.

As a consequence of the results in [1], we have

$$\mathbf{P}_H(\limsup_{u\to 0^+} \frac{W_u^H}{u^H\sqrt{\log\log u^{-1}}} = \sqrt{V_H}) = 1.$$

Hence it is impossible for W^H to have sample-paths Hölder continuous of an order greater than H. □

Let $W = \mathcal{C}_0([0,1],\mathbf{R})$ be the Banach space of continuous functions, null at time 0, equipped with the sup-norm and W^* be its topological dual. For any $H \in (0,1)$, $\mathbf{P}_H$ is the unique probability measure on W such that the canonical process $(W_s;\ s \in [0,1])$ is a centered Gaussian process with covariance kernel R_H :

$$\mathbf{E}_H[W_sW_t] = R_H(s,t).$$

The canonical filtration is given by $\mathscr{F}_t^H = \sigma\{W_s,\ s \le t\} \vee \mathcal{N}_H$ and $\mathcal{N}_H$ is the set of the $\mathbf{P}_H$–negligible events. Let $\mathscr{H}_H$ be the Cameron-Martin space associated with $(W, \mathbf{P}_H)$: the unique Hilbert space (identified with its dual) continuously and densely embedded in W such that, for any η in W^*,

$$\int_W e^{i<\eta,w>_{W^*,W}} d\mathbf{P}_H(w) = \exp\Big(-\|\tilde{\eta}\|^2_{\mathscr{H}_H}/2\Big), \tag{7}$$

where $\tilde{\eta}$ is the image of η under the injection $W^* \subset \mathscr{H}_H$. In order to be able to describe $\mathscr{H}_H$, we need the following preliminary lemma :

Lemma 3.1. *For any $H \in (0,1)$, $R_H(s,t)$ can be written as*

$$R_H(s,t) = \int_0^1 K_H(s,r)K_H(t,r)dr, \tag{8}$$

in operator notations, $R_H = K_HK_H^*$, where K_H is the Hilbert–Schmidt operator introduced in Theorem [2.1]. We hereafter identify an operator and its kernel.

Proof. For $H > 1/2$, it is easy to see that

$$R_H(s,t) = \frac{V_H}{4H(2H-1)} \int_0^t \int_0^s |r-u|^{2H-2}\, du\, dr$$

Moreover(see [2]),

$$\frac{V_H}{4H(2H-1)}|r-u|^{2H-2} = (ru)^{H-1/2} \int_0^{r\wedge u} v^{1/2-H}(r-v)^{H-3/2}(u-v)^{H-3/2}\, dv.$$

Hence for $H > 1/2$, (8) holds with

$$K_H(t,r) = \frac{r^{1/2-H}}{\Gamma(H-1/2)} \int_r^t u^{H-1/2}(u-r)^{H-3/2}\ du\ \mathbf{1}_{[0,t]}(r).$$

A change of variable in this equation transforms the integral term in

$$(t-r)^{H-1/2} r^{H-1/2} \int_0^1 u^{H-3/2}\Big(1-(1-t/r)u\Big)^{H-1/2}\ du.$$

By the definition (2) of hypergeometric functions, we see that (6) holds true for $H > 1/2$. Using property (4), we have

$$K_H(t,r) = \frac{2^{-2H}\sqrt{\pi}}{\Gamma(H)\sin(\pi H)} r^{H-1/2} + \frac{1}{2\Gamma(H+1/2)}(t-r)^{H-1/2} F(1/2-H, 1, 2-2H, \frac{r}{t}).$$

If $H < 1/2$ then the hypergeometric function of the latter equation is continuous with respect to r on $[0,t]$ because $2-2H-1-1/2+H = 1/2-H$ is positive. Hence, for $H < 1/2$, $K_H(t,r)(t-r)^{1/2-H} r^{1/2-H}$ is continuous with respect to r on $[0,t]$. For $H > 1/2$, the hypergeometric function is no more continuous in t but we have [11] :

$$F(1/2-H, 1, 2-2H, \frac{r}{t}) = C_1 F(1/2-H, 1, H+1/2, 1-r/t) + C_2(1-r/t)^{1/2-H}(r/t)^{2H-1}.$$

Hence, for $H \geq 1/2$, $K_H(t,r) r^{H-1/2}$ is continuous with respect to r on $[0,t]$. Fix $\delta \in [0,1/2)$ and $t \in (0,1]$, we have :

$$|K_H(t,r)| \leq C r^{-|H-1/2|}(t-r)^{-(1/2-H)_+}\mathbf{1}_{[0,t]}(r)$$

where C is uniform with respect to $H \in [1/2-\delta, 1/2+\delta]$. Thus, the two functions defined on $\{H \in \mathcal{C},\ |H-1/2| < 1/2\}$ by

$$H \in (0,1) \longmapsto R_H(s,t) \text{ and } H \in (0,1) \longmapsto \int_0^1 K_H(s,r)K_H(t,r)\ dr$$

are well defined, analytic with respect to H and coincide on $[1/2, 1)$, thus they are equal for any $H \in (0,1)$ and any s and t in $[0,1]$. □

In the previous proof we proved a result which is so useful in its own that it deserves to be a theorem :

Theorem 3.2. *For any $H \in (0,1)$, there exist a constant c_H such that for any t and r, we have :*

$$|K_H(t,r)| \le c_H r^{-|H-1/2|}(t-r)^{-(1/2-H)_+}\mathbf{1}_{[0,t]}(r), \tag{9}$$

where $x_+ = \max(x, 0)$.

Theorem 3.3. *1. $\mathscr{H}_H = \{K_H\dot{h};\ \dot{h} \in L^2([0,1], dt)\}$, i.e., any $h \in \mathscr{H}_H$ can be represented as*

$$h(t) = K_H\dot{h}(t) \overset{def}{=} \int_0^1 K_H(t,s)\dot{h}(s)\ ds,$$

where $\dot{h}$ belongs to $L^2([0,1])$. For any $\mathscr{H}_H$–valued random variable u, we hereafter denote by $\dot{u}$ the $L^2([0,1];\mathbf{R})$–valued random variable such that

$$u(w,t) = \int_0^t K_H(t,s)\dot{u}(w,s)\, ds.$$

2. The scalar product on $\mathscr{H}_H$ is given by

$$(h,g)_{\mathscr{H}_H} = (K_H\dot{h}, K_H\dot{g})_{\mathscr{H}_H} \overset{def}{=} (\dot{h},\dot{g})_{L^2([0,1])}.$$

3. The injection R_H from W^* into $\mathscr{H}_H$ can be decomposed as $R_H\eta = K_H(K_H^*\eta)$. Furthermore, the restriction of K_H^* to W^* is the injection from W^* into $L^2([0,1])$:

$$W^* @ > K_H^* >> L^2([0,1];\mathbf{R}) @ > K_H >> \mathscr{H}_H @ > i_H >> W.$$

Remark 3.1. Note that as a vector space, $\mathscr{H}_H$ is equal to $I_{0^+}^{H+1/2}(L^2([0,1]))$ but the norm on each of these spaces are different since the norm of an element h in the latter space is the L^2 norm of $I_{0^+}^{-H-1/2}(h)$

Proof. From Theorem [2.1], we know that K_H is a bijection from $L^2([0,1])$ onto $I_{0^+}^{H+1/2}(L^2([0,1])) \subset W$. For any $\alpha > -1/2$, $(K_H x^\alpha)(t) = c_{\alpha,H} t^{\alpha+H+1/2}$, hence $\mathscr{H}_H$ contains all the polynomials null at 0 so that $\mathscr{H}_H$ is dense in W from Stone–Weierstrass theorem.

Let i_H denote the inclusion from $\mathscr{H}_H$ into W, i_H^* the inclusion from W^* into $\mathscr{H}_H{}^*$, j_H the canonical identification isomorphism between $\mathscr{H}_H{}^*$ and $\mathscr{H}_H$ and $R_H = j_H \circ i_H^*$, i.e., R_H is the *embedding* of W^* into $\mathscr{H}_H$. For $\eta \in W^*$, we have on one hand

$$\Big(R_H(\eta), h\Big)_{\mathscr{H}_H} \overset{def}{=} \int_0^1 \dot{\overline{R_H(\eta)}}(s)\dot{h}(s)\ ds$$

and on the other hand,

$$\begin{aligned}\Big(R_H(\eta), h\Big)_{\mathscr{H}_H} =< \eta, i_H(h) >_{W^*,W} &= \int_0^1 \int_0^1 K_H(t,s)\dot{h}(s)\ ds\ \eta(dt)\\ &= \int_0^1 (K_H^*\eta)(s)\dot{h}(s)\ ds,\end{aligned}$$

where K_H^* is the adjoint of K_H for the $L^2([0,1])$ scalar-product. It follows that $\dot{\overline{R_H(\eta)}} = K_H^*\eta$ and then that $R_H = K_H \circ K_H^*$.

It remains to prove (7); for we compute the $\mathscr{H}_H$ norm of $R_H(\eta)$:

$$\begin{aligned}\int \langle \eta, w\rangle^2_{W^*,W} d\mathbf{P}_H(w) &= \mathbf{E}_H\left[\int_0^1 \int_0^1 W_s W_t\ \eta(ds)\eta(dt)\right]\\ &= \int_0^1\int_0^1 R_H(t,s)\eta(ds)\eta(dt)\\ &= \int_0^1\int_0^1\int_0^1 K_H(t,r)K_H(s,r)\ dr\ \eta(ds)\eta(dt)\\ &= \|K_H^*\eta\|^2_{L^2([0,1])} = \|R_H(\eta)\|^2_{\mathscr{H}_H}.\end{aligned}$$

Hence $\mathscr{H}_H$ has the three properties defining the Cameron–Martin space of the Wiener space $(W, \mathbf{P}_H)$. □

Corollary 3.1. *The fractional Brownian motion has the representation in law :*

$$W_t^H = \int_0^t K_H(t,s) dW_s^{1/2}.$$

Proof. The process $\left(\int_0^t K_H(t,s)dW_s^{1/2},\ t \in [0,1]\right)$ is Gaussian with the convenient covariance kernel. □

Remark 3.2. We will show in the sequel that this representation holds in the trajectorial sense with a fixed, standard Brownian motion constructed on $(W, \mathbf{P}_H)$. The representation of the corollary is different from the one in [10] since it requires only one standard Brownian motion instead of two.

The computer simulation of fBm paths is a classical problem of numerical analysis, the difficulty being due to the non–trivial correlation between the increments. The following proposition gives an approximation scheme :

Proposition 3.1. *Let π^n be an increasing sequence of partitions of $[0,1]$ such that the mesh $|\pi^n|$ of π^n tends to 0 as n goes to infinity. The sequence of processes $\{W^n,\ n \geq 0\}$ defined by*

$$W_t^n = \sum_{t_i^n \in \pi^n} \frac{1}{t_{i+1}^n - t_i^n} \int_{t_i^n}^{t_{i+1}^n} K_H(t,s)ds\ (W^{1/2}_{t_{i+1}^n} - W^{1/2}_{t_i^n})$$

converges to W^H in $L^2(\mathbf{P}_{1/2} \otimes ds)$.

Proof. Let $\mathcal{G}_n$ be the σ-field generated by $\{W^{1/2}_{t_i^n},\ t_i^n \in \pi^n\}$. For t fixed, the sequence

$$W_t^n \stackrel{def}{=} \mathbf{E}_{1/2}\left[\int_0^t K_H(t,s)dW_s^{1/2} \mid \mathcal{G}_n\right]$$

is a square integrable martingale with respect to the filtration $(\mathcal{G}_n)$. Moreover, using the Jensen inequality,

$$\sup_n \mathbf{E}_{1/2}\left[\int_0^1 (W_t^n)^2\ dt\right] \leq \int_0^1 R_H(t,t)\ dt < +\infty.$$

Hence the sequence $(W^n)_n$ is an $L^2([0,1])$–valued $(\mathcal{G}_n)$ square integrable martingale so that it converges $\mathbf{P}_H$ almost everywhere and in $L^2(W; L^2([0,1]))$, i.e.,

$$\lim_{n\to+\infty} \mathbf{E}_{1/2}\left[\int_0^1 (W_t^n - W_t)^2\ dt\right] = 0.$$

Furthermore, a simple calculation (using the Gaussian character and independence of the random variables $W^{1/2}_{t_{i+1}^n} - W^{1/2}_{t_i^n}$) shows that

$$\mathbf{E}_{1/2}\left[\int_0^t K_H(t,s)dW_s^{1/2}|\mathcal{G}_n\right] = \sum_{t_i^n \in \pi^n} \frac{1}{t_{i+1}^n - t_i^n} \int_{t_i^n}^{t_{i+1}^n} K_H(t,s)ds\ (W^{1/2}_{t_{i+1}^n} - W^{1/2}_{t_i^n})$$

which proves the result. □

4 Preliminaries and the Malliavin Calculus

For details on the construction of Malliavin calculus on Wiener spaces, we refer to [15, 17]. As for all Gaussian spaces, we have (see [4])

Theorem 4.1 (Cameron-Martin Theorem). *For any $R_H\eta \in \mathscr{H}_H$,*

$$\mathbf{E}_H\left[F(w+R_H\eta)\right] = \int F(w)\exp\Big(\langle \eta, w\rangle - \|R_H\eta\|^2_{\mathscr{H}_H}/2\Big)\, d\mathbf{P}_H(w) \tag{10}$$

Definition 4.1. Let X be a separable Hilbert space and $F\ : W \longrightarrow X$ an X-valued functional of the form

$$F(w) = \sum_{i=1}^{k} f_i(\langle l_1, w\rangle, \ldots, \langle l_n, w\rangle)\; x_i \tag{11}$$

where for each $i \in \{1, \ldots, n\}$, l_i is in W^* and x_i belongs to X. F is said to be a *smooth cylindric* functional (respectively a *polynomial*) when f_i is an element of the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ (respectively of the set of real polynomials with n variables). We will denote by $\mathcal{S}(X)$ (respectively $\mathcal{P}(X)$) the set of X-valued smooth cylindric functionals and simply $\mathcal{S}$ (respectively $\mathcal{P}$) when $X = \mathbf{R}$.

Consider $C_0^H = \mathbf{R}$ and for $n > 0$, define C_n^H as the closed vector space spanned in $L^2(\mathbf{P}_H)$ by the elements of $\mathcal{P}$ of degree less than n. Set $\mathcal{C}_0^H = C_0^H$ and suppose $\mathcal{C}_1^H, \ldots, \mathcal{C}_n^H$ are defined, then, we define $\mathcal{C}_{n+1}^H$ as the orthogonal complement of $\mathcal{C}_1^H \oplus \cdots \oplus \mathcal{C}_n^H$ in C_{n+1}^H. As for all Gaussian spaces, we have the chaos decomposition :

Theorem 4.2.

$$L^2(\mathbf{P}_H) = \bigoplus_{n\geq 0} \mathcal{C}_n^H.$$

This means that every $\mathbf{P}_H$-square integrable functional from W to $\mathbf{R}$ can be written in a unique way as

$$F = \sum_{n=0}^{+\infty} J_n^H F \tag{12}$$

where J_n^H is the orthogonal projection of $L^2(\mathbf{P}_H)$ onto $\mathcal{C}_n^H$.

Definition 4.2 (Ornstein-Uhlenbeck semi-group).
For $S \ni F : W \longrightarrow \mathbf{R}$ in $\cup_{p\geq 1} L^p(\mathbf{P}_H)$ and $t \geq 0$, we define $(T_t^H F)(w)$ by the Mehler formula :

$$T_t^H F(w) = \int F(e^{-t}w + \sqrt{1-e^{-2t}}y)\; d\mathbf{P}_H(y).$$

It turns out that for $F \in \mathcal{S}$, $(t \mapsto T_t^H F)$ is differentiable in L^p with respect to t, so that we can define the Ornstein-Uhlenbeck operator $\mathcal{L}_H$:

Definition 4.3 (Ornstein-Uhlenbeck operator)**.** $\mathcal{L}_H$ is defined on $\mathcal{S}$ by

$$\mathcal{L}_H F(w) = \frac{d}{dt} T_t F(w) \Big|_{t=0}.$$

The operator $\mathcal{L}_H$ can then be extended on $L^p(\mathbf{P}_H)$ as the infinitesimal generator of a contraction semi-group on $L^p(\mathbf{P}_H)$; let $\mathrm{Dom}_p(\mathcal{L}_H)$ be the domain of the extension of $\mathcal{L}_H$ in $L^p(\mathbf{P}_H)$.

Definition 4.4 (Sobolev spaces)**.** Let $p \geq 1$, q such that $p^{-1} + q^{-1} = 1$ and $k \in \mathbb{Z}$. $\mathbb{D}_{p,k}^H(X)$ is the completion of $\mathcal{S}(X)$ with respect to the norm

$$\|F\|_{p,k,H} \overset{def}{=} \|(I - \mathcal{L}_H)^{k/2} F\|_{L^p_H}$$

where

$$(I - \mathcal{L}_{\mathcal{H}})^{k/2} F = \sum_{n=0}^{+\infty} (1+n)^{k/2} J_n^H F.$$

It is well known that $\mathbb{D}_{p,k}^H(X)$ is the dual of $\mathbb{D}_{p,-k}^H$ with the duality pairing

$$\langle F, G \rangle = \mathbf{E}_H \left[((I - \mathcal{L})^{k/2} F,\ (I - \mathcal{L})^{-k/2} G)_X \right]$$

and that for any $k' \leq k$,

$$\mathbb{D}_{p,k}^H(X) \subset \mathbb{D}_{p,k'}^H(X) \subset \mathbb{D}_{p,0}^H = L^p(W; X) \subset \mathbb{D}_{p,-k'}^H(X) \subset \mathbb{D}_{p,-k}^H(X).$$

We also introduce the spaces $\mathbb{D}_{p,k,a}^H(\mathscr{H}_H)$ which are for any $p \geq 1$ and any $k \in \mathbf{N}$ the subspaces of $\mathbb{D}_{p,k}^H(\mathscr{H}_H)$ composed by the adapted processes. For any $k \in -\mathbf{N}$, $\mathbb{D}_{p,k,a}^H(\mathscr{H}_H)$ is the dual of $\mathbb{D}_{p,-k,a}^H(\mathscr{H}_H)$. The notation $\mathbb{D}_\infty^H(X)$ (respectively $\mathbb{D}_{\infty,a}^H(\mathscr{H}_H)$) will stand for $\cap_{p,k\geq 0} \mathbb{D}_{p,k}^H(X)$ (respectively $\cap_{p,k\geq 0} \mathbb{D}_{p,k,a}^H(\mathscr{H}_H)$) and $\mathbb{D}_{-\infty}^H(X) = \cup_{p,-k\geq 0} \mathbb{D}_{p,k}^H(X)$, respectively $\mathbb{D}_{-\infty,a}^H(\mathscr{H}_H) = \cup_{p,-k\geq 0} \mathbb{D}_{p,k,a}^H(\mathscr{H}_H)$.

Definition 4.5. For F in $\mathcal{S}(X)$, the H-Gross-Sobolev derivative of F, denoted by ∇F and is the $\mathscr{H}_H \otimes X$–valued mapping defined by

$$\nabla F(w) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\langle l_1, w\rangle, \ldots, \langle l_n, w\rangle)\ R_H(l_i) \otimes x. \tag{13}$$

Remark 4.1. Take $l_i = \varepsilon_{t_i}$, the Dirac measure at time t_i and let

$$\begin{aligned} F(w) &= f(\langle \varepsilon_{t_1}, w\rangle, \dots, \langle \varepsilon_{t_n}, w\rangle) \\ &= f(W_{t_1}, \dots, W_{t_n}), \end{aligned}$$

then we have by (13),

$$\nabla F(w) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) R_H(\varepsilon_{t_i}),$$

i.e.,

$$\nabla F(w)(s) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) R_H(t_i, s).$$

Starting with $X = \mathbf{R}$, we define the derivative of a real valued smooth functional F, then, taking $X = \mathscr{H}_H{}^{\otimes n-1}$, we can define, inductively, the n-th derivative of F by $\nabla^{(n)} F = \nabla(\nabla^{(n-1)} F)$. The directional derivative of $F \in \mathcal{S}(X)$ in the direction $R_H \eta \in \mathscr{H}_H$ is given by :

$$(\nabla F, R_H \eta)_{\mathscr{H}_H} = \frac{d}{dt} F(w + t.R_H \eta)\Big|_{t=0} \tag{14}$$

and from the Cameron–Martin theorem, we see that ∇F depends only on the equivalence classes with respect to $\mathbf{P}_H$ and

$$\mathbf{E}_H \left[(\nabla F, R_H \eta)_{\mathscr{H}_H} \right] = \mathbf{E}_H \left[F \langle w, \eta \rangle \right]$$

which implies the closability of ∇ and its iterates. We can also define Sobolev spaces using the operator ∇ and its iterates as in the finite dimensional case, this definition is equivalent to Definition [4.4]; this is due to the following inequalities of P.A. Meyer : for $p > 1$ and $k \in \mathbb{Z}$, there exist $a_{p,k,H}$ and $A_{p,k,H}$ such that, for every $F \in S$,

$$\begin{aligned} a_{p,k,H} \| |\nabla^{(k)} F|_{HS} \|_{L^p(\mathbf{P}_H)} &\le \|F\|_{p,k,H} \\ &\le A_{p,k,H} \Big(\|F\|_{L^p(\mathbf{P}_H)} + \| |\nabla^{(k)} F|_{HS} \|_{L^p(\mathbf{P}_H)} \Big), \end{aligned} \tag{15}$$

where $|\nabla^{(k)} F|_{HS}$ stands for the Hilbert-Schmidt norm of $\nabla^{(k)} F$: if $\{\eta_n,\, n \in \mathbf{N}\}$ is an orthonormal basis of $\mathscr{H}_H{}^{\otimes k} \otimes X$,

$$|\nabla^{(k)} F|_{HS}^2 = \sum_{n=0}^{\infty} \Big(\nabla^{(k)} F, \eta_n \Big)^2_{\mathscr{H}_H{}^{\otimes k} \otimes X}.$$

As a consequence, ∇ can be extended as a continuous linear operator from $\mathbb{D}^H_{p,k}(X)$ to $\mathbb{D}^H_{p,k-1}(\mathscr{H}_H \otimes X)$ for any $p > 1$ and $k \in \mathbb{Z}$.

Definition 4.6 (Divergence or Skohorod integral)**.** Let δ_H be the formal adjoint of ∇ with respect to $\mathbf{P}_H$: $\forall F \in \mathcal{S}, \forall u \in \mathcal{S}(\mathscr{H}_H)$,

$$\mathbf{E}_H\left[F\ \delta_H u\right] = \mathbf{E}_H\left[\Big(\nabla F,\ u\Big)_{\mathscr{H}_H}\right]. \tag{16}$$

Since ∇ has continuous extensions, δ_H has a continuous linear extension from $\mathbb{D}^H_{p,k}(\mathscr{H}_H)$ to $\mathbb{D}^H_{p,k-1}$ for any $p > 1$ and any $k \in \mathbb{N}$.

Remark 4.2. For $F \in \mathcal{S}(W^*)$ of the form (11) with $k = 1$, the divergence of F is defined by

$$\begin{aligned}(\delta_H F)(w) &= f(\langle l_1, w\rangle, \ldots, \langle l_n, w\rangle)\langle x, w\rangle_{W^*,W} \\ &\qquad - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\langle l_1, w\rangle, \ldots, \langle l_n, w\rangle) < R_H(l_i), R_H(x) >_{\mathscr{H}_H}.\end{aligned}$$

Take $x = \epsilon_t,\ f = 1$, we get

$$W_t = \delta_H\Big(R_H(\epsilon_t)\Big) = \delta_H\Big(K_H(K_H(t, .))\Big).$$

Moreover, for $\eta \in W^*$, we have $\delta_H(R_H\eta) = \langle \eta, w\rangle_{W^*,W}$ almost surely. Furthermore, we have

$$\mathbf{E}_H\left[F\ \delta_H u\right] = \mathbf{E}_H\left[(\nabla F,\ u)_{\mathscr{H}_H}\right]$$

for any $\mathbb{D}^H_{p,k+1},\ u \in \mathbb{D}^H_{q,-k}(\mathscr{H}_H)$ for any $p > 1$, $p^{-1} + q^{-1} = 1$ and $k \in \mathbb{Z}$.

We recall several identities valid in any Wiener space. Let $F, F_1, \ldots, F_n$ be in $\mathbb{D}_\infty$, G_1, G_2 in $\mathbb{D}_\infty(\mathscr{H}_H)$, f in the Schwartz space of $\mathbf{R}^n$ and $T \in \mathbb{D}_{-\infty}(\mathscr{H}_H)$,

$$\begin{aligned}\nabla f(F_1, \ldots, F_n) &= \sum_{i=1}^{n} \partial_i f(F_1, \ldots, F_n)\ \nabla F_i, \\ \delta_H(\nabla F) &= -\mathcal{L}_H F, \\ \delta_H(FT) &= F\delta_H T - \nabla_T F, \qquad (17)\\ \mathbf{E}_H\left[\delta_H(G_1)\delta_H(G_2)\right] &= \mathbf{E}_H\left[(G_1, G_2)_{\mathscr{H}_H}\right] + \mathbf{E}_H\left[\operatorname{trace}(\nabla G_2 \circ \nabla G_1)\right], \qquad (18)\\ \nabla_{G_2}\Big(\delta_H G_1\Big) &= (G_1, G_2)_{\mathscr{H}_H} + \delta_H(\nabla_{G_2} G_1) + \operatorname{trace}(\nabla G_2 \circ \nabla G_1). \qquad (19)\end{aligned}$$

Proposition 4.1. *For $u \in \mathscr{H}_H$, let $\Lambda^u_1 \stackrel{def}{=} \exp(\delta_H u - 1/2\|u\|^2_{\mathscr{H}_H})$, we have*

$$J^H_n \Lambda^u_1 = \frac{1}{n!}\delta^{(n)}_H u^{\otimes n}.$$

More generally, for $F \in \cup_{k\in\mathbb{Z}}\mathbb{D}^H_{2,k}$,

$$J^H_n F = \frac{1}{n!}\delta^{(n)}_H\Big(\mathbf{E}_H\left[\nabla^{(n)}F\right]\Big).$$

Proof. By the definition (14) of ∇, we have for any polynomial G on W, λ,

$$\begin{aligned}\mathbf{E}_H\left[G(w+\lambda u)\right] &= \sum_{n=0}^{+\infty}\frac{\lambda^n}{n!}\mathbf{E}_H\left[(\nabla^{(n)}G, u^{\otimes n}\Big)_{\mathscr{H}_H{}^{\otimes n}}\right]\\ &= \sum_{n=0}^{+\infty}\frac{\lambda^n}{n!}\mathbf{E}_H\left[G.\delta^{(n)}_H u^{\otimes n}\right].\end{aligned}$$

On the other hand, by the Cameron–Martin Theorem 4.1,

$$\mathbf{E}_H\left[G(w+\lambda u)\right] = \sum_{k=0}^{+\infty}\mathbf{E}_H\left[GJ^H_k\Big(\exp(\lambda\delta_H u - \frac{\lambda^2}{2}\|u\|^2_{\mathscr{H}_H})\Big)\right].$$

Using the generating functions of Hermite polynomials (see e.g. [11]), we see that

$$\exp(\lambda\delta_H u - \frac{\lambda^2}{2}\|u\|^2_{\mathscr{H}_H}) = \sum_{n=0}^{+\infty}H_n\Big(\frac{\delta_H u}{\sqrt{2}\|u\|_{\mathscr{H}_H}}\Big)\frac{\|u\|^n_{\mathscr{H}_H}}{\sqrt{2}^n n!}\lambda^n,$$

where H_n is the n-th Hermite polynomial. Since $H_n(\frac{\delta_H u}{\sqrt{2}\|u\|_{\mathscr{H}_H}})$ belongs to $\mathcal{C}_n$,

$$\mathbf{E}_H\left[G(w+\lambda u)\right] = \sum_{k=0}^{+\infty}\mathbf{E}_H\left[G.J^H_k(F)\right]\lambda^k.$$

The result follows by identification of the two power series.

This result can also be written

$$J^H_n\Lambda^u_1 = \frac{1}{n!}\delta^{(n)}_H\Big(\mathbf{E}_H\left[\nabla^{(n)}\Lambda^u_1\right]\Big).$$

Since the linear combinations of Wick exponentials (i.e., exponentials of the form $\exp(\delta_H h - \frac{1}{2}\|h\|^2_{\mathscr{H}_H})$) are dense in $\mathbb{D}^H_\infty$ and J^H_n is a continuous operator, the result follows by density for any $F \in \cap_{k\in\mathbb{N}}\mathbb{D}^H_{2,k}$. Now by duality the formula also holds for F in the dual of the latter space, i.e., for $F \in \cup_{k\in\mathbb{Z}}\mathbb{D}^H_{2,k}$. □

4.1 Relations between δ_H and other stochastic integrals

As for any Gaussian process, the Wiener integral with respect to the fractional Brownian motion is usually defined as the linear extension from $\mathscr{H}_H$ in $L^2(\mathbf{P}_H)$ of the isometric map :

$$d_w \;:\; R_H(t_i,.) \longmapsto W_{t_i}.$$

Since $K_H(L^2([0,1])) = \mathscr{H}_H$, one could also define a Wiener integral for deterministic integrand belonging to $L^2([0,1])$ by considering the linear extension from $L^2([0,1])$ to $L^2(\mathbf{P}_H)$ of the isometry :

$$\partial_w \ : \ K_H(t_i, .) \longmapsto W_{t_i}. \tag{20}$$

When $H = 1/2$, $K_{1/2}(t, .) = \mathbf{1}_{[0,t]}$ so that we have :

$$\partial_w(h) = \lim_{n \to +\infty} \sum_{i=0}^{2^n - 1} h_{i2^{-n}} (W_{(i+1)2^{-n}} - W_{i2^{-n}}), \tag{21}$$

for any continuous h. From (20), it is clear that (21) does not hold any more when $H \neq 1/2$. Thus there exist at least two different approaches to define a stochastic integral with respect to the fractional Brownian motion : one approach consists of using the Skohorod integral which is defined for any Gaussian process, the second approach uses Riemann sums similar to the right–hand–side of (21)– see [3, 6, 9]. The resulting integrals will have more or less similar properties to stochastic integrals defined with respect to semi–martingales, however none of them will be completely adequate to construct a full stochastic calculus.

1. We define stochastic integral of first type as :

   $$\int_0^1 \dot{u}_s \, \delta_H W_s \overset{def}{=} \delta_H(K_H \dot{u})$$

   for any $\dot{u}$ such that $K_H \dot{u} \in \mathbb{D}^H_{-\infty}(\mathscr{H}_H)$.

2. For $H > 1/2$, if we do not identify $\mathscr{H}_H$ and its dual but $L^2([0,1])$ and its dual, we have the following diagram :

   $$\begin{array}{ccccccccc} W^* & @>i_H^*>> & \mathscr{H}_H{}^* & @>K_H^*>> & L^2([0,1]) & @>K_H>> & \mathscr{H}_H & @>i_H>> & W \\ & & \bigcup & & \| & & \bigcap & & \\ W^* & @>i_{1/2}^*>> & \mathscr{H}_{1/2}{}^* & @>K_{1/2}^*>> & L^2([0,1]) & @>K_{1/2}>> & \mathscr{H}_{1/2} & @>i_{1/2}>> & W \end{array}$$

   Thus it is meaningful to define a stochastic integral of second type by :

   $$\int_0^1 \dot{u}_s \, \overset{\circ}{d} W_s = \int_0^1 (\mathcal{K}_H^* \dot{u})(s) \, \delta_H W_s = \delta_H(K_H \mathcal{K}_H^* \dot{u}),$$

   where $\mathcal{K}_H = K_{1/2}^{-1} K_H$, for $\dot{u}$ such that $K_H \mathcal{K}_H^* \dot{u} \in \mathbb{D}^H_{-\infty}(\mathscr{H}_H)$.

Proposition 4.2. *When $H > 1/2$, the stochastic integral of second type coincides with the stochastic integral defined by Riemann sums. We have the following identity provided that $\dot{u}$ is deterministic and both sides exist :*

$$\int_0^1 \dot{u}_s \overset{\circ}{d}W_s = \lim_{|\pi_n|\to 0} \sum_{t_i^n \in \pi_n} \dot{u}_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n}). \tag{22}$$

When $\dot{u} \in \mathbb{D}_{2,1}^H(L^2([0,1]))$ *and* $\operatorname{trace}(\nabla K_H \mathcal{K}_H^* \dot{u})$ *is well defined, we have :*

$$\int_0^1 \dot{u}_s \overset{\circ}{d}W_s = \lim_{|\pi_n|\to 0} \sum_{t_i^n \in \pi_n} \dot{u}_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n}) - \operatorname{trace}(\nabla K_H \mathcal{K}_H^* \dot{u}). \tag{23}$$

Proof. Note that $K_{1/2}^*\varepsilon_t = \mathbf{1}_{[0,t]}$ and $K_H^*\varepsilon_t = K_H(t,.)$, thus $\mathcal{K}_H^*(\mathbf{1}_{[0,t]}) = K_H(t,.)$. If $\dot{u}$ is of the form

$$\dot{u}(s) = \sum_{i=1}^n \dot{u}_i \mathbf{1}_{(t_i,t_{i+1}]}(s),$$

with $\dot{u}_i \in \mathbb{D}_{2,1}^H$, we have by (17),

$$\begin{aligned}
\delta_H(K_H\mathcal{K}_H^*\dot{u}) &= \sum_{i=1}^n \delta_H\Big(\dot{u}_i K_H(K_H(t_{i+1},.) - K_H(t_i,.))\Big) \\
&= \sum_{i=1}^n \dot{u}_i \delta_H(K_H(K_H(t_{i+1},.) - K_H(t_i,.))) \\
&\qquad - \int_0^1 \dot{\nabla}_s \dot{u}_i (K_H(t_{i+1},s) - K_H(t_i,s))\,ds \\
&= \sum_{i=1}^n \dot{u}_i(W_{t_{i+1}} - W_{t_i}) - trace(\nabla K_H \circ \mathcal{K}_H^* \dot{u}).
\end{aligned}$$

Since $\nabla u = 0$ when u is deterministic, the two results follow by a limiting procedure when both sides of the last relation converge. □

In view of (23), it seems sensible to define a Stratonovitch integral by :

$$\int_0^1 \dot{u}_s \tilde{d}W_s = \int_0^1 \dot{u}_s \overset{\circ}{d}W_s + \operatorname{trace}(\nabla K_H \mathcal{K}_H^* \dot{u}),$$

for any $\dot{u}$ such that the right–hand–side of the last equation is meaningful. Note that as it is shown in (23),

$$\int_0^1 \dot{u}_s \tilde{d}W_s = \lim_{|\pi_n|\to 0} \sum_{t_i^n \in \pi_n} \dot{u}_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n}).$$

We will see below in the Itô formula that the Stratonovitch integral for $H \neq 1/2$ does not behave as nicely as it does when $H = 1/2$.

One can still have an integration by parts with $\overset{\circ}{d}W$ if we use a damped Sobolev derivative,

Proposition 4.3. *When $H > 1/2$, set*

$$D\psi \overset{def}{=} K_H\Big(\mathcal{K}_H(\dot{\nabla}_{.}\psi)\Big).$$

We have :

$$\mathbf{E}_H\left[\int_0^1 \dot{u}_s \overset{\circ}{d}W_s.\psi\right] = \mathbf{E}_H\left[(K_H\dot{u}, D\psi)_{\mathscr{H}_H}\right], \tag{24}$$

and

$$Df(W_{t_1},\dots,W_{t_n})(s) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W_{t_1},\dots,W_{t_n})\int_0^s K_H(s,r)\frac{\partial R_H(t_i,r)}{\partial r}\,dr.$$

Note that $\operatorname{trace}(\nabla K_H\mathcal{K}_H^*\dot{u}) = \operatorname{trace}(DK_H\dot{u})$, *when both of the traces exist.*

Proof. For any convenient u and ψ,

$$\begin{aligned}
\mathbf{E}_H\left[\int_0^1 \dot{u}_s \overset{\circ}{d}W_s.\psi\right] &= \mathbf{E}_H\left[\int_0^1 (\mathcal{K}_H^*\dot{u})(s)\,\delta_H W_s.\psi\right]\\
&= \mathbf{E}_H\left[\int_0^1 (\mathcal{K}_H^*\dot{u})(s)\dot{\nabla}_s\psi\,ds\right]\\
&= \mathbf{E}_H\left[\int_0^1 \dot{u}(s)\mathcal{K}_H(\dot{\nabla}_{.}\psi)(s)\,ds\right]\\
&= \mathbf{E}_H\left[(K_H\dot{u}, D\psi)_{\mathscr{H}_H}\right].
\end{aligned}$$

The proof of the second part of the proposition is simply the application of the definitions of D and ∇. □

Definition 4.7. For any $H \in (0,1)$, we define the family $\{\pi_t^H,\ t \in [0,1]\}$ of orthogonal projections in $\mathscr{H}_H$ by

$$\pi_t^H(K_H u) \overset{def}{=} K_H(u\mathbf{1}_{[0,t]}),\ u \in L^2([0,1]).$$

The second quantization $\Gamma(\pi_t^H)$ of π_t^H is an operator from $L^2(\mathbf{P}_H)$ into itself defined by,

$$F = \sum_{n\geq 0}\delta_H^{(n)} f_n \longmapsto \Gamma(\pi_t^H)(F) \overset{def}{=} \sum_{n\geq 0}\delta_H^{(n)}\Big((\pi_t^H)^{\otimes n} f_n\Big).$$

From [4.1], we have, for $u \in \mathscr{H}_H$,

$$\Gamma(\pi_t^H)\Big(\Lambda_1^u\Big) = \exp(\delta_H(\pi_t^H u) - \frac{1}{2}\|\pi_t^H u\|^2_{\mathscr{H}_H}) \overset{def}{=} \Lambda_t^u.$$

The bijectivity of the operator K_H has the following consequence :

Theorem 4.3. $\mathscr{F}_t^H = \sigma\{\delta_H(\pi_t^H u),\ u \in \mathscr{H}_H\} \vee \mathscr{N}_H$.

Proof. The definition of $\mathscr{F}_t^H$ says that it is equal to the completion of the σ-field generated by random variables of the form $\delta_H\Big(K_H(K_H(s,.))\Big)$ with $s \le t$. This amounts to say that

$$\mathscr{F}_t^H = \sigma\{\delta_H(K_H u),\ u \in \operatorname{span}\{K_H(s,.),\ s \le t\}\} \vee \mathscr{N}_H.$$

Observe now that we can replace $\operatorname{span}\{K_H(s,.),\ s \le t\}$ by its closure in $L^2([0,t])$ without changing $\mathscr{F}_t^H$. Actually, if u is the limit in $L^2([0,1])$ of a sequence $(u_n)_n$ of elements of $\operatorname{span}\{K_H(s,.),\ s \le t\}$ then $K_H u_n$ converges to $K_H u$ in $\mathscr{H}_H$ and thus $\delta_H(K_H u_n)$ tends to $\delta_H(K_H u)$ in $L^2(\mathbf{P}_H)$. Hence $\delta_H(K_H u)$ belongs to $\mathscr{F}_t^H$ and the two σ-fields are thus equal. Now it turns out that $\operatorname{span}\{K_H(s,.),\ s \le t\}$ is total in $L^2([0,t])$: if g belongs to the orthogonal complement of this space in $L^2([0,t])$:

$$0 = (g, K_H(s,.))_{L^2([0,t])} = K_H g(s) \text{ for all } s \le t,$$

so that $g \equiv 0$ in $L^2([0,t])$. The proof is finished by observing that the image $L^2([0,t])$ by K_H is nothing but the space $\pi_t^H(\mathscr{H}_H)$. □

Theorem 4.4. *For any F in $L^2(\mathbf{P}_H)$,*

$$\Gamma(\pi_t^H)F = \mathbf{E}_H\left[F \mid \mathscr{F}_t^H\right],$$

in particular,

$$\mathbf{E}_H\left[W_t \mid \mathscr{F}_r^H\right] = \int_0^t K_H(t,s)\mathbf{1}_{[0,r]}(s)\,\delta_H W_s,\ \textit{and}$$
$$\mathbf{E}_H\left[\exp(\delta_H u - 1/2\|u\|^2_{\mathscr{H}_H}) \mid \mathscr{F}_t^H\right] = \exp(\delta_H \pi_t^H u - 1/2\|\pi_t^H u\|^2_{\mathscr{H}_H}),$$

for any $u \in \mathscr{H}_H$.

Proof. Let $\{h_n,\ n \ge 1\}$ be a denumerable family of elements of $\mathscr{H}_H$ and let $V_n = \sigma\{\delta_H h_k,\ 1 \le k \le n\}$. Denote by π_n the orthogonal projection on $\operatorname{span}\{h_1, \ldots, h_n\}$. For any f bounded, for any $u \in \mathscr{H}_H$, by the Cameron–

Martin theorem we have

$$\begin{aligned}
&\mathbf{E}_H\left[\Lambda_1^u f(\delta_H h_1,\dots,\delta_H h_n)\right]\\
&= \mathbf{E}_H\left[f(\delta_H h_1(w+u),\dots,\delta_H h_n(w+u))\right]\\
&= \mathbf{E}_H\left[f(\delta_H h_1+(h_1,u)_{\mathscr{H}_H},\dots,\delta_H h_n+(h_n,u)_{\mathscr{H}_H})\right]\\
&= \mathbf{E}_H\left[f(\delta_H h_1(w+\pi_n u),\dots,\delta_H h_n(w+\pi_n u))\right]\\
&\qquad\qquad\qquad\qquad = \mathbf{E}_H\left[\Lambda_1^{\pi_n u} f(\delta_H h_1,\dots,\delta_H h_n)\right],
\end{aligned}$$

hence

$$\mathbf{E}_H\left[\Lambda_1^u \,|\, V_n\right] = \Lambda_1^{\pi_n u}. \tag{25}$$

Choose h_n of the form $\pi_t^H(e_n)$ where $\{e_n,\, n\geq 1\}$ is an orthonormal basis of $\mathscr{H}_H$, i.e., $\{h_n,\, n\geq 1\}$ is an orthonormal basis of $\pi_t^H(\mathscr{H}_H)$. By the previous theorem, $\bigvee_n V_n = \mathscr{F}_t^H$ and it is clear that π_n tends pointwise to π_t^H, hence from (25) and martingale convergence theorem, we can conclude that

$$\mathbf{E}_H\left[\Lambda_1^u \,|\, \mathscr{F}_t^H\right] = \Lambda_1^{\pi_t^H u} = \Lambda_t^u.$$

Moreover, for $u\in\mathscr{H}_H$,

$$\Gamma(\pi_t^H)(\Lambda_1^u) = \Lambda_1^{\pi_t^H u},$$

hence by density of linear combinations of Wick exponentials, for any $F\in L^2(\mathbf{P}_H)$,

$$\Gamma(\pi_t^H)F = \mathbf{E}_H\left[F \,|\, \mathscr{F}_t^H\right],$$

and the proof is completed. □

Theorem 4.5 (cf [16]). *Let F be $\mathbb{D}_{2,1}^H$, F belongs to $\mathscr{F}_t^H$ iff $\nabla F = \pi_t^H \nabla F$.*

Proof. Let F be a $\mathscr{F}_t^H$–measurable element of $\mathbb{D}_{2,1}^H$, $\{u_n^t,\ n\geq 0\}$ be an orthonormal basis of $L^2([0,t])$ and V_n^t be the σ field generated by $\{\delta_H K_H u_i^t,\ i\leq n\}$. Since $\vee_n V_n^t = \mathscr{F}_t^H$, the sequence $F_n = \mathbf{E}_H\left[F|\ V_n^t\right]$ converges to F in $\mathbb{D}_{2,1}^H$. Since $\pi_t^H K_H u_n^t = K_H u_n^t$, for F_n we have $\nabla F_n = \pi_t^H \nabla F_n$ and $\nabla F = \pi_t^H \nabla F$ follows. In the converse direction, remark that it is sufficient to prove that $T_s F$ is $\mathscr{F}_t^H$-measurable for any $s>0$, where $(T_s)_s$ is the Ornstein–Uhlenbeck semi–group (see Definition [4.2]). It is easy to see that

$$\nabla T_s F = e^{-s} T_s \nabla F = e^{-s} T_s \pi_t \nabla F = \pi_t e^{-s} T_s \nabla F = \pi_t \nabla T_s F.$$

Iterating this relation, we obtain

$$\nabla^{(n)} T_s F = (\pi_t^H)^{\otimes n} \nabla^{(n)} T_s F.$$

Moreover, from the Wiener chaos expansion, we have, for any $u \in \mathscr{H}_H$,

$$\begin{aligned} E[T_s F(w+u)] &= \sum_{n=0}^{\infty} \frac{1}{n!} (E[\nabla^{(n)} T_s F], u^{\otimes n})_{\mathscr{H}_H{}^{\otimes n}} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\pi_t^{H \otimes n} E[\nabla^{(n)} T_s F], u^{\otimes n})_{\mathscr{H}_H{}^{\otimes n}} = E[T_s F(w + \pi_t^H u)]. \end{aligned}$$

On the other hand, from the Cameron–Martin formula,

$$\begin{aligned} &\mathbf{E}_H \left[T_s F \exp(\delta_H u - \frac{1}{2} \|u\|^2_{\mathscr{H}_H}) \right] = \mathbf{E}_H \left[T_s F(w+u) \right] \\ &= \mathbf{E}_H \left[T_s F(w + \pi_t u) \right] = \mathbf{E}_H \left[T_s F \cdot \mathbf{E}_H \left[\exp(\delta_H u - \frac{1}{2} \|u\|^2_{\mathscr{H}_H}) \ \mid \mathscr{F}_t^H \right] \right] \end{aligned}$$

and this completes the proof since the linear combinations of Wick exponentials are dense in all the spaces $\mathbb{D}^H_{p,k}$. □

Proposition 4.4. *A process $u(w,t)$ is $\{\mathscr{F}_t^H,\, t \in [0,1]\}$–adapted iff its* density *process $\dot{u}$, i.e.,*

$$u(w,t) = \int_0^t K_H(t,s) \dot{u}(w,s) \ ds$$

is $\{\mathscr{F}_t^H,\, t \in [0,1]\}$–adapted.

Proof. It is obvious that adaptedness of $\dot{u}$ adapted entails that u is adapted. In the converse direction, note that $\dot{u}(w,t) = (K_H^{-1} u(w,.))(t)$ and from Theorem [2.1] we see that all the quadratures involved in the computation of $(K_H^{-1} u)(t)$ have their support in $[0,t]$. Hence, u adapted entails $\dot{u}$ adapted. □

Theorem 4.6 (Itô–Clark representation formula)**.** *For any $F \in \mathbb{D}^H_{2,1}$,*

$$\begin{aligned} F - \mathbf{E}_H[F] &= \int_0^1 \mathbf{E}_H \left[K_H^{-1} (\nabla F)(s) |\ \mathscr{F}_s^H \right] \ \delta_H W_s \\ &= \delta_H \Big(K_H (\mathbf{E}_H \left[K_H^{-1} (\nabla F)(.) | \mathscr{F}_. \right]) \Big). \end{aligned}$$

Proof. From the chaos expansion (4.1), we know that for any $u \in \mathscr{H}_H$,

$$\Lambda_1^u = 1 + \int_0^1 \dot{u}_s \Lambda_s^u \, \delta_H W_s \, .$$

Moreover, linear combinations of Wick exponentials are dense in $L^2(\mathbf{P}_H)$ hence it is sufficient to prove that for any $u \in \mathscr{H}_H$ any centered F,

$$\mathbf{E}_H\left[F\Lambda_1^u\right] = \mathbf{E}_H\left[\int_0^1 \mathbf{E}_H\left[K_H^{-1}(\nabla F)(s) \,|\, \mathscr{F}_s^H\right] \,\delta_H W_s.\delta_H\Big(K_H(u_.\Lambda_.^u)\Big)\right].$$

Integrating by parts and since $\{\Lambda_s^u,\, s \in [0,1]\}$ is an adapted process, we have :

$$\begin{aligned}
&\mathbf{E}_H\left[\int_0^1 \mathbf{E}_H\left[K_H^{-1}(\nabla F)(s) \,|\, \mathscr{F}_s^H\right] \,\delta_H W_s.\delta_H\Big(K_H(u_.\Lambda_.^u)\Big)\right]\\
&= \mathbf{E}_H\left[\int_0^1 \mathbf{E}_H\left[K_H^{-1}(\nabla F)(s) \,|\, \mathscr{F}_s^H\right] u_s\Lambda_s^u\, ds\right]\\
&= \mathbf{E}_H\left[\int_0^1 \mathbf{E}_H\left[K_H^{-1}(\nabla F)(s) u_s\Lambda_s^u \,|\, \mathscr{F}_s^H\, ds\right]\right]\\
&= \mathbf{E}_H\left[\int_0^1 K_H^{-1}(\nabla F)(s) u_s\Lambda_s^u\, ds\right] = \mathbf{E}_H\left[F.\delta_H\Big(K_H(u_.\Lambda_.^u)\Big)\right]\\
&\qquad\qquad = \mathbf{E}_H\left[F(\Lambda_1^u - 1)\right] = \mathbf{E}_H\left[F\Lambda_1^u\right],
\end{aligned}$$

as it was required. □

Theorem 4.7. *For any u adapted and in $L^2(W;\mathscr{H}_H)$, the process $\{M_t = \delta_H(\pi_t^H u),\, t \in [0,1]\}$ is a square integrable martingale,i.e.,*

$$\mathbf{E}_H\left[\delta_H u \,|\, \mathscr{F}_t^H\right] = \delta_H(\pi_t^H u) = \int_0^t K_H^{-1}u(s)\,\delta_H W_s,$$

whose Doob–Meyer process is $t \mapsto \int_0^t u_s^2\, ds$. In particular, for $v \in L^2(W; L^2([0,1]))$ adapted,

$$t \longmapsto \int_0^t v(s)\,\delta_H W_s \text{ is a martingale.}$$

Proof. Let u be adapted and belong to $\mathbb{D}_\infty(\mathscr{H}_H)$, for any $v \in \mathscr{H}_H$, since π_t^H is a projector,

$$\begin{aligned}
\mathbf{E}_H\left[\mathbf{E}_H\left[\delta_H u \,|\, \mathscr{F}_t^H\right]\Lambda_t^v\right] &= \mathbf{E}_H\left[\delta_H u.\Lambda_t^v\right]\\
&= \mathbf{E}_H\left[(u, \nabla\Lambda_t^v)_{\mathscr{H}_H}\right] = \mathbf{E}_H\left[\left(u, \pi_t^H v\right)_{\mathscr{H}_H}\Lambda_t^v\right]\\
&= \mathbf{E}_H\left[\left(\pi_t^H u, \pi_t^H v\right)_{\mathscr{H}_H}\Lambda_t^v\right] = \mathbf{E}_H\left[\delta_H(\pi_t^H u)\Lambda_t^v\right].
\end{aligned}$$

Moreover, for any $v \in \mathscr{H}_H$ such that $v = (\mathrm{Id}_{\mathscr{H}_H} - \pi_t^H)v$, we have

$$\begin{aligned}
\left(\nabla\delta_H\pi_t^H u, v\right)_{\mathscr{H}_H} &= \left(\pi_t^H u, (\mathrm{Id}_{\mathscr{H}_H} - \pi_t^H)v\right)_{\mathscr{H}_H}\\
&\quad + \delta_H\big(\left(\nabla\pi_t^H u, (\mathrm{Id}_{\mathscr{H}_H} - \pi_t^H)v\right)_{\mathscr{H}_H}\big) = 0,
\end{aligned}$$

since by Theorem [4.5], $\nabla\pi_t^H u = \pi_t^H \nabla \pi_t^H u$ and π_t^H is an orthogonal projection. By density, we see that

$$\mathbf{E}_H\left[\delta_H u \,|\, \mathscr{F}_t^H\right] = \delta_H(\pi_t^H u),$$

for any u in $\mathbb{D}^H_{-\infty,a}(\mathscr{H}_H)$.

For any $t \in [0,1]$ and any $\phi \in \mathbb{D}_\infty^H$,

$$\begin{aligned}
&\mathbf{E}_H\left[\left(\delta_H(\pi_t^H u)^2 - \|\pi_t^H u\|^2_{\mathscr{H}_H}\right)\phi\right] \\
&= \mathbf{E}_H\left[\left(\pi_t^H u, \nabla(\delta_H(\pi_t^H u\phi))\right)_{\mathscr{H}_H}\right] - \mathbf{E}_H\left[\|\pi_t^H u\|^2_{\mathscr{H}_H}\phi\right] \\
&= \mathbf{E}_H\left[\left(\pi_t^H u, \nabla\phi\right)_{\mathscr{H}_H}\right] + \mathbf{E}_H\left[\delta_H\left(\nabla_{\pi_t^H u}\pi_t^H u\right)\phi\right] \\
&= \mathbf{E}_H\left[\left(\nabla^{(2)}\phi, \pi_t^H u \otimes \pi_t^H u\right)_{\mathscr{H}_H\otimes\mathscr{H}_H}\right] + \mathbf{E}_H\left[\left(\nabla\left(\pi_t^H u, \nabla\phi\right)_{\mathscr{H}_H}, \pi_t^H u\right)_{\mathscr{H}_H}\right].
\end{aligned}$$

Let $0 \le s \le t \le 1$ and ϕ be an $\mathscr{F}_s^H$–measurable element of $\mathbb{D}_\infty^H$. Theorem [4.5] implies that $(\nabla\phi, v)_{\mathscr{H}_H} = \left(\nabla\phi, \pi_s^H v\right)_{\mathscr{H}_H}$ for any v. Hence,

$$\begin{aligned}
&\mathbf{E}_H\left[\left(\delta_H(\pi_t^H u)^2 - \|\pi_t^H u\|^2_{\mathscr{H}_H}\right)\phi\right] \\
&= \mathbf{E}_H\left[\left(\nabla^{(2)}\phi, \pi_s^H u \otimes \pi_s^H u\right)_{\mathscr{H}_H\otimes\mathscr{H}_H}\right] + \mathbf{E}_H\left[\left(\nabla\left(\pi_s^H u, \nabla\phi\right)_{\mathscr{H}_H}, \pi_s^H u\right)_{\mathscr{H}_H}\right] \\
&\qquad = \mathbf{E}_H\left[\left(\delta_H(\pi_s^H u)^2 - \|\pi_s^H u\|^2_{\mathscr{H}_H}\right)\phi\right],
\end{aligned}$$

thus $\{\delta_H(\pi_t^H u)^2 - \|\pi_t^H u\|^2_{\mathscr{H}_H},\, t \in [0,1]\}$ is a martingale. □

As a corollary we have a *constructive* proof of the Levy–Hida representation theorem :

Theorem 4.8 (Levy–Hida representation)**.**

1. *The process*
$$B = \{B_t \overset{def}{=} \delta_H(\pi_t^H K_H 1),\, t \in [0,1]\}$$
is a $\mathbf{P}_H$*–standard Brownian motion whose filtration is equal to* $\{\mathscr{F}_t^H,\, t \in [0,1]\}$.

2. *If we denote by* dB *the Itô integral with respect to this Brownian motion, we have for any adapted* u, $\mathbf{P}_H$*–almost surely,*
$$\int_0^t u_s\, \delta_H W_s = \int_0^t u_s\, dB_s, \text{ for all } t.$$

Proof. B is a Gaussian process whose covariance kernel is given (see Corollary [4.7]) by

$$\mathbf{E}_H\left[B_s B_t\right] = (\pi_s K_H 1, \pi_t K_H 1)_{\mathscr{H}_H} = \int_0^1 \mathbf{1}_{[0,t]}(u)\mathbf{1}_{[0,s]}(u)\, du = \min(s,t).$$

The equality of the filtrations follows simply from Proposition [4.3].

Let u be of the form

$$u(w,s) = \sum_{i=1}^n u_i \mathbf{1}_{(t_i,t_{i+1}]}(s),$$

where for any i, u_i is square–integrable and $\mathscr{F}^H_{t_i}$–measurable. If for any i, u_i belongs to $\mathbb{D}^H_{2,1}$ we have by (17)

$$\int_0^1 u_s\, \delta_H W_s = \sum_{i=1}^n u_i(B_{t_{i+1}} - B_{t_i}) - \int_0^1 \dot{\nabla}_r u_i \mathbf{1}_{(t_i,t_{i+1}]}(r)\, dr.$$

Since u_i is $\mathscr{F}^H_{t_i}$–measurable, the last integrand is zero by Theorem [4.5]. By a limiting procedure,

$$\int_0^1 u_s\, \delta_H W_s = \sum_{i=1}^n u_i(B_{t_{i+1}} - B_{t_i}),$$

even when u_i is only square integrable and in $\mathscr{F}^H_{t_i}$. Moreover, by Theorem [4.7],

$$\mathbf{E}_H\left[(\int_0^1 u_s\, \delta_H W_s)^2\right] = \mathbf{E}_H\left[\int_0^1 u_s^2\, ds\right].$$

On the other hand, the Itô stochastic integral of u with respect to B is by definition given by :

$$\sum_{i=1}^n u_i(B_{t_{i+1}} - B_{t_i}).$$

By continuity of δ_H and dB, it follows that the stochastic integrations with respect to $\delta_H W$ or to dB coincide on the set of adapted processes which belong to $L^2(W;\mathscr{H}_H)$. □

The classical martingales characterization says that :

Corollary 4.1. *Every* $(\mathbf{P}_H, \{\mathscr{F}^H_t,\ t \in [0,1]\})$ *square integrable martingale* M *can be written as*

$$M_0 + \delta_H\Big(\pi^H_t u\Big),$$

where

$$u_t = \mathbf{E}_H\left[\nabla M_1 \,|\, \mathscr{F}^H_t\right].$$

Theorem 4.9 (Girsanov theorem)**.** *Let $u = K_H \dot{u}$ be an adapted process in $L^2(W; \mathscr{H}_H)$ such that*

$$\mathbf{E}_H\left[\Lambda_1^u\right] = 1, \tag{26}$$

and let $\mathbf{P}_u$ be the probability defined by

$$\left.\frac{d\mathbf{P}_u}{d\mathbf{P}_H}\right|_{\mathscr{F}_t^H} = \Lambda_t^u = \exp\Big(\delta_H \pi_t^H u - \frac{1}{2}\|\pi_t^H u\|^2_{\mathscr{H}_H}\Big).$$

The law of the process

$$\{W_t - \int_0^t K_H(t,s)\, ds,\ t \in [0,1]\} \text{ under } \mathbf{P}_u$$

is the same as the law of the canonical process W under $\mathbf{P}_H$. In other words, for any $v = K_H \dot{v}$ adapted and in $L^2(W; \mathscr{H}_H)$, the law of the process

$$\{\int_0^t K_H(t,s)\dot{v}_s\, \delta_H W_s - \int_0^t K_H(t,s)\dot{u}_s \dot{v}_s\, ds,\ t \in [0,1]\} \text{ under } \mathbf{P}_u$$

is the same as the law of the process

$$\{\int_0^t K_H(t,s)\dot{v}_s\, \delta_H W_s,\ t \in [0,1]\} \text{ under } \mathbf{P}_H.$$

Proof. Fix $(t_1, \ldots, t_n)$ in $[0,1]^n$ and consider the n–dimensional martingale

$$Z^n :\ r \longmapsto \Big(\int_0^{t_i \wedge r} K_H(t_i, s)\dot{v}_s\, \delta_H W_s\Big)_{i=1}^n.$$

The classical Girsanov theorem stands that the $\mathbf{P}_u$–law of the process

$$r \mapsto \Big(\int_0^{t_i \wedge r} K_H(t_i, s)\dot{v}_s\, \delta_H W_s - \int_0^{t_i \wedge r} K_H(t_i, s)\dot{u}_s \dot{v}_s\, ds\Big)_{i=1}^n$$

is the same as the $\mathbf{P}_H$–law of $\{Z_r^n,\ r \in [0,1]\}$. Hence for any bounded measurable f from $\mathbf{R}^n$ to $\mathbf{R}$,

$$\mathbf{E}_{\mathbf{P}_u}\left[f(\ldots, \int_0^{t_i \wedge r} K_H(t_i, s)\dot{v}_s\, \delta_H W_s - \int_0^{t_i \wedge r} K_H(t_i, s)\dot{u}_s \dot{v}_s\, ds, \ldots)\right]$$
$$= \mathbf{E}_H\left[f(\ldots, \int_0^{t_i \wedge r} K_H(t_i, s)\dot{v}_s\, \delta_H W_s, \ldots)\right],$$

for any $r \in [0,1]$. The result follows by taking $r = 1$ in the last equation. □

Since we have reduced the Girsanov problem of fBm to that of the ordinary Brownian motion, we can make the full use of the usual Novikov condition to ensure the uniform integrability of Λ^u. Namely, it is sufficient that

$$\mathbf{E}_H\left[\exp\frac{1}{2}\|u\|^2_{\mathscr{H}_H}\right] < +\infty$$

for (26) to hold.

Application : Consider the process X defined by

$$X_t = \theta t + W_t^H$$

and let $\mathbf{P}_X$ denote its law. We aim to estimate θ through the observation of a sample-path of X over $[0,t]$. Let ϕ be such that $(K_H\phi)(t) = t$, by the Cameron-Martin Theorem and since for deterministic u, $(\delta_H u)(w + K_H\phi) = (\delta_H u)(w) + (u, K_H\phi)_{\mathscr{H}_H}$

$$\begin{aligned}
\left.\frac{d\mathbf{P}_X}{d\mathbf{P}_H}\right|_{\mathscr{F}_t^H}(w) &= \mathbf{E}_H\left[\exp(\theta\delta_H(K_H\phi)(w) - \frac{\theta^2}{2}\|\phi\|^2_{L^2})|\ \mathscr{F}_t^H\right] \\
&= \exp(\theta\delta_H(\pi_t^H K_H\phi)(w) - \frac{\theta^2}{2}\|\pi_t^H\phi\|^2_{L^2}), \text{ or} \\
\left.\frac{d\mathbf{P}_H}{d\mathbf{P}_X}\right|_{\mathscr{F}_t^H}(w) &= \exp(-\theta\delta_H(\pi_t^H K_H\phi)(w) + \frac{\theta^2}{2}\|\pi_t^H\phi\|^2_{L^2}), \text{ i.e.,} \\
\mathbf{E}_H\left[F(w)\right] = \mathbf{E}_H&\left[F(X(w))\exp(-\theta\delta_H(\pi_t^H K_H\phi)(X(w)) + \frac{\theta^2}{2}\|\pi_t^H\phi\|^2_{L^2})\right].
\end{aligned}$$

Hence the $\mathbf{P}_H$ maximum likelihood ratio estimate $\hat{\theta}_t$ of θ is

$$\hat{\theta}_t = \frac{1}{\|\pi_t^H\phi\|^2_{\mathscr{H}_H}}\delta_H(\pi_t^H K_H\phi)(X(w)).$$

From Theorem [2.1], we have

$$\phi(s) \stackrel{def}{=} \frac{\Gamma(3/2 - H)}{\Gamma(2-2H)} s^{1/2-H}.$$

Hence,

$$\hat{\theta}_t = \frac{2^{2-2H}\Gamma(2-H)}{\sqrt{\pi}\ t^{2-2H}}\delta_H(\pi_t^H K_H s^{1/2-H})(X(w)).$$

5 The Itô Formula

Hereafter we assume that H is greater than $1/2$. In this case, the fractional Brownian motion has a zero quadratic variation hence it is a Dirichlet process. It is well known that for such processes there exists an Itô formula (see

[7]) of the form :

$$F(W_t) = F(W_0) + \int_0^t F'(W_s)\, dW_s, \tag{27}$$

where the stochastic integral dW_s is defined as the limit of Riemann sums as in formula (22)and F is of class $\mathcal{C}^2$. In fact, we improve, in our situation, the results known for Dirichlet processes in the sense that a somewhat explicit expression of the right–hand–side of (27) is given. The processes of the form (28) have been chosen because they constitute a class which is stable with respect to absolutely continuous changes of probability measures – see Theorem [4.9].

Let $\rho(s) = s^{1-2H}$ and denote by $\mathscr{H}_{H,\rho}$ the image of $L^2_\rho([0,1]) = \{h : [0,1] \to \mathbf{R},\ \int_0^1 h^2(s)\rho(s)\, ds < +\infty\}$ under K_H. Note that when $H > 1/2$, $L^2_\rho([0,1]) \subset L^2([0,1])$ because $\rho(s) \geq 1$ for any $s \in [0,1]$. The space $\mathscr{H}_{H,\rho}$ is endowed with the norm induced by $L^2_\rho([0,1])$, i.e.,

$$\|u\|_{\mathscr{H}_{H,\rho}} = \|K_H^{-1} u\|_{L^2_\rho}.$$

The weighted Sobolev space $\mathbb{D}^H_{2,k,\rho}$ is the set of elements F of $\mathbb{D}^H_{2,k}$ such that for any $n \leq k$, the n-th Gross-Sobolev derivative of F belongs to $\mathscr{H}_{H,\rho}^{\otimes n}$ and the norm of F in this space is defined by

$$\|F\|^2_{2,k,H,\rho} = \|F\|^2_{L^2(\mathbf{P}_H)} + \sum_{i=1}^k \mathbf{E}_H\left[\|\nabla^{(i)} F\|^2_{\mathscr{H}_{H,\rho}^{\otimes i}}\right].$$

Before the proof, we give two lemmas for later use.

Lemma 5.1. *Let $u(w,s)$ be such that $\mathbf{E}_H\left[\int_0^1 |u(w,s)|\ ds\right] < +\infty$ then $t \to \mathbf{E}_H\left[I^\alpha_{0^+} u(t)\right]$ is continuous.*

Proof. Since $I^\alpha_{0^+}$ is a positive linear operator,

$$|\mathbf{E}_H\left[I^\alpha_{0^+} u(t+\varepsilon) - I^\alpha_{0^+} u(t)\right]| \leq (I^\alpha_{0^+}\mathbf{E}_H\left[|u|\right])(t+\varepsilon) - (I^\alpha_{0^+}\mathbf{E}_H\left[|u|\right])(t).$$

Hence by the remark following Proposition [2.1], the right–hand–side of the last inequality converges to 0 as ε tends to 0. □

Lemma 5.2. *Let $(Y_\varepsilon)_\varepsilon$ be a process which converges to 0 in $L^2(\mathbf{P}_H)$ when ε tends to 0. Let $h(w,u)$ be in $L^2(L^2([0,1]))$. Then*

$$\lim_{\varepsilon\to 0} \mathbf{E}_H\left[\varepsilon^{-1} Y_\varepsilon \int_t^{t+\varepsilon} h(u)\ du\right] = 0\ dt\ a.s..$$

Proof. Using the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathbf{E}_H\left[\varepsilon^{-1}Y_\varepsilon\int_t^{t+\varepsilon}h(u)\,du\right]^2 \le \\
&\quad 2\mathbf{E}_H\left[Y_\varepsilon^2\right]\{\mathbf{E}_H\left[\Big(\int_t^{t+\varepsilon}h(u)-h(t)\,\frac{du}{\varepsilon}\Big)^2\right]+2\mathbf{E}_H\left[h(t)^2\right]\} \\
&\quad \le 2\mathbf{E}_H\left[Y_\varepsilon^2\right]\left\{\varepsilon^{-1}\int_t^{t+\varepsilon}\mathbf{E}_H\left[(h(u)-h(t))^2\right]\,du+2\mathbf{E}_H\left[h(t)^2\right]\right\}.
\end{aligned}$$

The second term of the previous sum is finite for almost every t and the first converges to 0 as ε tends to 0, hence all of the right–hand–side converges to 0 for almost every t. □

Theorem 5.1. *Suppose $H > 1/2$, let $F : \mathbf{R} \longmapsto \mathbf{R}$ be twice differentiable with bounded derivatives and X be a process of the form*

$$\begin{aligned}
X_t(w) &= X_0(w) + (K_H\xi(w))(t) + \delta_H\Big(K_H(\sigma(w,.)K_H(t,.))\Big)(w) \qquad (28)\\
&= X_0(w) + \int_0^t K_H(t,s)\xi(w,s)ds + \int_0^t K_H(t,s)\sigma(w,s)\delta_H W_s,
\end{aligned}$$

where X_0 belongs to $\mathbb{D}^H_{2,1,\rho}$, ξ is in $\mathbb{D}^H_{2,1,\rho}(L^2_\rho([0,1]))$ and σ is in $\mathbb{D}^H_{2,2,\rho}(L^2_\rho([0,1]))$. We have

$$\begin{aligned}
F(X_t) - F(X_0) &= \int_0^t I_{t^-}^{H-1/2}\Big(F'(X_u)u^{H-1/2}\Big)(s)s^{1/2-H}\xi(s)\,ds \\
&+ \int_0^t s^{1/2-H}\sigma(s)I_{t^-}^{H-1/2}\Big(u^{H-1/2}F'(X_u)\Big)(s)\,\delta_H W_s \\
&+ \int_0^t I_{t^-}^{H-1/2}\Big(F''(X_u)u^{H-1/2}\Big)(s)s^{1/2-H}\sigma(s)\dot{\nabla}_s X_0\,ds \\
&+ \int_0^t I_{t^-}^{H-1/2}\Big(F''(X_u)u^{H-1/2}K_H(\dot{\nabla}_s\xi)(u)\Big)(s)s^{1/2-H}\sigma(s)\,ds \\
&+ \int_0^t I_{t^-}^{H-1/2}\Big(F''(X_u)u^{H-1/2}K_H(u,s)\Big)(s)s^{1/2-H}\sigma(s)^2\,ds \\
&+ \int_0^t I_{t^-}^{H-1/2}\Big(F''(X_u)u^{H-1/2}\int_0^1\dot{\nabla}_s\sigma(r)K_H(u,r)\delta_H W_r\Big)(s)s^{1/2-H}\sigma(s)\,ds. \qquad (29)
\end{aligned}$$

Proof. Let us give the main idea of the proof : as a first step we shall suppose that X_0, ξ, σ are smooth processes in the sense that they have bounded Sobolev–Gross derivatives which are continuous with respect to all their parameters, i.e.,$\dot{\nabla}_u X_0$, $\dot{\nabla}_u\sigma(s)$, $\dot{\nabla}_u\xi(s)$ and $\ddot{\nabla}_{u,v}\sigma(s)$ are bounded and continuous with respect to w, u, v and s. Then we use the fundamental theorem *of differential calculus which says that*

$$\mathbf{E}_H\left[F(X_t)\psi\right] = \mathbf{E}_H\left[F(X_0)\psi\right] + \int_0^t \frac{d}{ds}\mathbf{E}_H\left[F(X_s)\psi\right]\,ds,$$

where ψ *is in* $\mathbb{D}_\infty$. *Afterwards, we rewrite the above identity using the stochastic integration by parts formula* (16) *and Fubini's theorem to obtain the claimed expression.*

1) In a first step, we assume that we have the following additional hypothesis : F is $\mathcal{C}^2$ with bounded derivatives, $\dot\nabla_u X_0,\ \dot\nabla_u\sigma(s),\ \dot\nabla_u\xi(s)$ and $\ddot\nabla_{u,v}\sigma(s)$ are bounded and continuous with respect to w, u, v and s. Let ψ be in $\mathbb{D}_\infty^H$ with bounded Gross-Sobolev derivatives,

$$\begin{aligned}&\mathbf{E}_H\left[\Big(F(X_{t+\varepsilon}) - F(X_t)\Big)\psi\right] = \mathbf{E}_H\left[F'(X_t)(X_{t+\varepsilon}-X_t)\psi\right]\\ &+\mathbf{E}_H\left[(X_{t+\varepsilon}-X_t)^2\int_0^1 F''(uX_t+(1-u)X_{t+\varepsilon})(1-u)\ du.\psi\right] = A_1 + A_2.\end{aligned}$$

Let us first consider A_1 :

$$\begin{aligned}A_1 &= \mathbf{E}_H\left[F'(X_t)\psi(K_H\xi(t+\varepsilon) - K_H\xi(t))\right]\\ &\quad + \mathbf{E}_H\left[F'(X_t)\psi\delta_H\Big(K_H(\sigma\{K_H(t+\varepsilon,.) - K_H(t,.)\})\Big)\right] = B_1 + B_2.\end{aligned}$$

Since ξ is bounded, $t \longmapsto (K_H\xi)(t)$ is differentiable for each w and

$$(K_H\xi)'(t) = t^{H-1/2} I_{0^+}^{H-1/2}\Big(u^{1/2-H}\xi(u)\Big)(t).$$

Hence, by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}B_1 = \mathbf{E}_H\left[F'(X_t)t^{H-1/2}I_{0^+}^{H-1/2}\Big(u^{1/2-H}\xi(u)\Big)(t).\psi\right]. \tag{30}$$

Because of the regularity of $X_0,\ \xi$ and σ, it is clear that X_t has a Gross-Sobolev derivative for every t and moreover (cf. (19)) :

$$\begin{aligned}\dot\nabla_u X_t = \dot\nabla_u X_0 &+ \int_0^t K_H(t,s)\dot\nabla_u\xi(s)\ ds\\ &\qquad + \sigma(u)K_H(t,u) + \int_0^t \dot\nabla_u\sigma(s)K_H(t,s)\delta_H W_s.\end{aligned}$$

Hence, we can write

$$\begin{aligned}B_2 &= \mathbf{E}_H\left[\int_0^1 \sigma(u)(K_H(t+\varepsilon,u) - K_H(t,u))\dot\nabla_u(F'(X_t)\psi)du\right]\\ &= \mathbf{E}_H\left[K_H\Big(\sigma(.)\dot\nabla_.(F'(X_t)\psi)\Big)(t+\varepsilon) - K_H\Big(\sigma(.)\dot\nabla_.(F'(X_t)\psi)\Big)(t)\right].\end{aligned}$$

The additional hypothesis imply that $t \longmapsto K_H\Big(\sigma(.)\dot{\nabla}_.(F'(X_t)\psi)\Big)(t)$ is continuously differentiable and by the dominated convergence theorem, we have :

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}B_2 = \mathbf{E}_H\left[t^{H-1/2}I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)\dot{\nabla}_u(F'(X_t)\psi)\Big)(t)\right].$$

Consider now the second-order term. Since F'' is bounded

$$A_2 \simeq c\mathbf{E}_H\left[(X_{t+\varepsilon}-X_t)^2.\psi\right].$$

Furthermore, by the reasoning used to obtain(30),

$$\mathbf{E}_H\left[(K_H\xi(t+\varepsilon)-K_H\xi(t))^2\right] = o(\varepsilon^2)$$

so that, it is sufficient to look at the term involving the stochastic integral :

$$\begin{aligned}
&\mathbf{E}_H\left[F''(X_t)\psi\delta_H\Big(K_H(\sigma\{K_H(t+\varepsilon,.)-K_H(t,.)\})\Big)^2\right]\\
&= \mathbf{E}_H\left[F''(X_t)\psi\Big(\int_0^t \sigma(u)(K_H(t+\varepsilon,u)-K_H(t,u)\Big)\delta_H W_u\right.\\
&\qquad\qquad\qquad\qquad\left. - \int_t^{t+\varepsilon}\sigma(u)K_H(t+\varepsilon,u)\delta_H W_u\Big)^2\right].\\
&= \mathbf{E}_H\left[F''(X_t)\psi\Big(\int_0^t\sigma(u)(K_H(t+\varepsilon,u)-K_H(t,u))\ \delta_H W_u\Big)^2\right]\\
&+ \mathbf{E}_H\left[F''(X_t)\psi\Big(\int_t^{t+\varepsilon}\sigma(u)K_H(t+\varepsilon,u)\ \delta_H W_u\Big)^2\right]\\
&+ 2\mathbf{E}_H\left[F''(X_t)\psi\Big(\int_0^t\sigma(u)(K_H(t+\varepsilon,u)-K_H(t,u))\ \delta_H W_u\right.\\
&\qquad\qquad\qquad\qquad\left.\times\int_t^{t+\varepsilon}\sigma(u)K_H(t+\varepsilon,u)\ \delta_H W_u\Big)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad = C_1+C_2+2C_3.
\end{aligned}$$

By (18) and the Cauchy-Schwarz inequality, we can upperbound C_1 by

$$\begin{aligned}
|C_1| &\le \|F''\|_\infty\|\psi\|_\infty\mathbf{E}_H\left[\int_0^{t+\varepsilon}\sigma(u)^2(K_H(t+\varepsilon,u)-K_H(t,u))^2\ du\right]\\
&+ \|F''\|_\infty\|\psi\|_\infty\mathbf{E}_H\left[\int_0^t\int_0^t\dot{\nabla}_u\sigma(s)\dot{\nabla}_s\sigma(u)(K_H(t+\varepsilon,u)-K_H(t,u))\right.\\
&\qquad\qquad\qquad\qquad\left.(K_H(t+\varepsilon,s)-K_H(t,s))\ du\ ds\quad\right]\\
&\qquad\qquad \le C\int_0^t(K_H(t+\varepsilon,u)-K_H(t,u))^2\ du = CV_H\varepsilon^{2H}.
\end{aligned}$$

By the same way, there exists a constant c such that

$$|C_3| \le c\int_t^{t+\varepsilon} (K_H(t+\varepsilon,u) - K_H(t,u))K_H(t+\varepsilon,u)\,du \\ + c\Big(\int_0^t (K_H(t+\varepsilon,u) - K_H(t,u))\,du\Big)\Big(\int_t^{t+\varepsilon} K_H(t+\varepsilon,u)\,du\Big).$$

As to C_2, we have

$$|C_2| \le c\mathbf{E}_H\left[\int_t^{t+\varepsilon} \sigma(u)^2 K_H(t+\varepsilon,u)^2\,du\right] \\ + c\mathbf{E}_H\left[\int_t^{t+\varepsilon}\int_t^{t+\varepsilon} \dot{\nabla}_s\sigma(u)\dot{\nabla}_u\sigma(s)K_H(t+\varepsilon,u)K_H(t+\varepsilon,s)\,du\,ds\right],$$

with another constant c. When divided by ε, the right–hand–side of the last equation goes to 0 because $K_H(t,t) = 0$. We have proved so far that, under the additional smoothness hypothesis, we have

$$\frac{d}{dt}\mathbf{E}_H\left[F(X_t)\psi\right] = \mathbf{E}_H\left[F'(X_t)t^{H-1/2}I_{0^+}^{H-1/2}\Big(u^{1/2-H}\xi(u)\Big)(t).\psi\right] \\ + \mathbf{E}_H\left[t^{H-1/2}F'(X_t)I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)\dot{\nabla}_u\psi\Big)(t)\right] \\ + \mathbf{E}_H\left[t^{H-1/2}F''(X_t)I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)\dot{\nabla}_u X_0\Big)(t).\psi\right] \\ + \mathbf{E}_H\left[t^{H-1/2}F''(X_t)I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)\int_0^t K_H(t,r)\dot{\nabla}_u\xi(r)dr\Big)(t).\psi\right] \\ + \mathbf{E}_H\left[t^{H-1/2}F''(X_t)I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)^2K_H(t,u)\Big)(t).\psi\right] \\ +\mathbf{E}_H\left[t^{H-1/2}F''(X_t)I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)\int_0^1 \dot{\nabla}_u\sigma(s)K_H(t,s)\delta_H W_s\Big)(t).\psi\right]. \tag{31}$$

2) The second step is to prove that each of these terms is continuous with respect to t in order to be able to integrate them over a finite interval. The first three terms are easily handled using Lemma [5.1]. We only give here the complete proof for the last term of the previous sum because it is the most difficult one, fourth and fifth terms are handled similarly : since $t^{H-1/2}F''(X_t)\psi$ is bounded, it is sufficient to prove that

$$t \longrightarrow \mathbf{E}_H\left[I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)\int_0^1 \dot{\nabla}_u\sigma(s)K_H(t,s)\delta_H W_s\Big)(t)\right]$$

is continuous. Let us define $Y_{t,u} = \int_0^1 \dot{\nabla}_u \sigma(s) K_H(t,s) \delta_H W_s$. We have :

$$
\begin{aligned}
&|\mathbf{E}_H \left[I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)Y_{t+\varepsilon,u}\Big)(t+\varepsilon) - I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)Y_{t,u}\Big)(t)\right]| \\
&\qquad \leq |\mathbf{E}_H \left[I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)(Y_{t+\varepsilon,u} - Y_{t,u})\Big)(t+\varepsilon)\right]| \\
&+ |\mathbf{E}_H \left[I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)Y_{t,u}\Big)(t+\varepsilon) - I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)Y_{t,u}\Big)(t)\right]|.
\end{aligned}
\tag{32}
$$

By (17),

$$
\begin{aligned}
&\mathbf{E}_H \left[(Y_{t+\varepsilon,u} - Y_{t,u})^2\right] \leq \mathbf{E}_H \left[\int_0^1 (\dot{\nabla}_u \sigma(s))^2 (K_H(t+\varepsilon,s) - K_H(t,s))^2 \; ds\right] \\
&+ \mathbf{E}_H \Bigg[\iint |\ddot{\nabla}_{v,u}\sigma(s)\ddot{\nabla}_{s,u}\sigma(v)|(K_H(t+\varepsilon,s) - K_H(t,s)) \\
&\qquad\qquad (K_H(t+\varepsilon,v) - K_H(t,v)) \; ds \; dv\Bigg] \\
&\leq C\int_0^1 (K_H(t+\varepsilon,s) - K_H(t,s))^2 ds \\
&\qquad \leq C(R_H(t+\varepsilon,t+\varepsilon) + R_H(t,t) - 2R_H(t+\varepsilon,t)) \leq C\varepsilon^{2H-1},
\end{aligned}
$$

where C is a constant independent of u. Hence,

$$
\begin{aligned}
&|\mathbf{E}_H \left[I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)(Y_{t+\varepsilon,u} - Y_{t,u})\Big)(t+\varepsilon)\right]| \\
&\qquad\qquad \leq C I_{0^+}^{H-1/2}(u^{1/2-H})(t+\varepsilon).\varepsilon^{H-1/2}.
\end{aligned}
$$

As to the last term of (32), using [5.1], it can be made as small as desired since

$$
\int_0^1 u^{1/2-H}\mathbf{E}_H\left[\sigma(u)Y_{t,u}\right] \; du \leq C\|\sigma\|_\infty \int_0^1 u^{1/2-H} R_H(t,u)^{1/2} \; du \; < +\infty.
$$

Now, before we can relax the additional hypothesis, we apply the Fubini Theorem (to exchange expectations and integrals) and the fractional integration by parts (5). For instance, for the second term of the right hand side sum, we obtain

$$
\begin{aligned}
&\mathbf{E}_H \left[\int_0^t s^{H-1/2}F'(X_t) I_{0^+}^{H-1/2}\Big(u^{1/2-H}\sigma(u)\dot{\nabla}_u\psi\Big)(s) \; ds\right] \\
&= \mathbf{E}_H \left[\int_0^t I_{t^-}^{H-1/2}(u^{H-1/2}F'(X_u))(s) s^{1/2-H}\sigma(s)\dot{\nabla}_s\psi \; ds\right] \\
&\qquad\qquad = \mathbf{E}_H \left[\psi \int_0^t s^{1/2-H}\sigma(s) I_{t^-}^{H-1/2}(u^{H-1/2}F'(X_u))(s) \; \delta_H W_s \;\right].
\end{aligned}
$$

The other terms are transformed similarly. If we denote by RHS the right hand side of (29), we know that

$$\mathbf{E}_H \left[(F(X_t) - F(X_0) - RHS)\psi \right] = 0,$$

for any ψ in $\mathbb{D}^H_\infty$ with bounded derivatives. Since both of $F(X_t) - F(X_0)$ and RHS belong to $L^2(\mathbf{P}_H)$, we deduce by the density of $\mathbb{D}^H_\infty$ in $L^2(\mathbf{P}_H)$ that $F(X_t) - F(X_0) = RHS$.

3) The third step is to prove that the additional hypothesis can be relaxed. Actually, our aim is to show that given X_0^n, σ^n, ξ^n and F_n converging respectively to X_0, σ, ξ and F in the respective normed spaces, the sequence $F_n(X_t^n) - F_n(X_0^n) - RHS(F_n, X^n)$ converges in L^1_H to $F(X_t) - F(X_0) - RHS(F, X)$. For this it is sufficient to show that $\mathbf{E}_H \left[RHS(F,X) \right]$ can be bounded by a polynomial in $\|F'\|_\infty$, $\|F''\|_\infty$, $\|X_0\|_{2,1,H,\rho}$, $\|\xi\|_{\mathbb{D}^H_{2,1,\rho}(L^2_\rho([0,1]))}$ and $\|\sigma\|_{\mathbb{D}^H_{2,2,\rho}(L^2_\rho([0,1]))}$. For instance, for the last term (29), we have

$$\begin{aligned}
&\mathbf{E}_H \left[\int_0^t I_{t^-}^{H-1/2} \Big(F''(X_u) u^{H-1/2} \int_0^1 \dot{\nabla}_s \sigma(r) K_H(u,r) \delta_H W_r \Big)(s) s^{1/2-H} \sigma(s) \; ds \right] \\
&\quad \le \mathbf{E}_H \left[\int_0^t (s^{1/2-H}\sigma(s))^2 \; ds^{1/2} \right] \\
&\times \mathbf{E}_H \left[\Big(\int_0^t I_{t^-}^{H-1/2} (u^{H-1/2} F''(X_u) \int_0^1 \dot{\nabla}_s \sigma(r) K_H(u,r) \delta_H W_r)(s) \Big)^2 \; ds \right]^{1/2} \\
&\quad \le C \|F''\|_\infty \|\sigma\|_{L^2(L^2_\rho)} \\
&\qquad \times \mathbf{E}_H \left[\int_0^t \Big(\int_s^t \int_0^u \dot{\nabla}_s \sigma(r) K_H(u,r) \; \delta_H W_r \; (u-s)^{H-3/2} \; du \Big)^2 \; ds \right]^{1/2}.
\end{aligned}$$

By the Jensen inequality applied to the measure $(u-s)^{H-3/2}\mathbf{1}_{[s,t]}(u)du$ (whose total mass is $(H-1/2)^{-1}(t-s)^{H-1/2}$), we upperbound the last expectation by a constant times the following integral :

$$\begin{aligned}
&\int_0^t (t-s)^{H-1/2}\int_s^t \mathbf{E}_H\left[\Big(\int_0^u \dot\nabla_s\sigma(r)K_H(u,r)\ \delta_H W_r\Big)^2\right](u-s)^{H-3/2}\ du\ ds\\
&\le \mathbf{E}_H\left[\int_0^t (t-s)^{H-1/2}\int_s^t\int_0^1 (\dot\nabla_s\sigma(r)r^{1/2-H})^2 dr\ du\ ds\right]\\
&\quad+\mathbf{E}_H\left[\int_0^t (t-s)^{H-1/2}\int_s^t\int_0^1\int_0^1 |\ddot\nabla_{v,s}\sigma(r)|^2\rho(r)\rho(v)\ dr\ dv\ du\ ds\right]\\
&\le \mathbf{E}_H\left[\int_0^1\int_0^1 \Big(\dot\nabla_s\sigma(r)(sr)^{1/2-H}\Big)^2 dr\ ds\right]\\
&\qquad+\mathbf{E}_H\left[\int_0^1\int_0^1\int_0^1 |\ddot\nabla_{v,s}\sigma(r)|^2\rho(r)\rho(v)\rho(s))\ dr\ dv\ ds\right]\le C\|\sigma\|^2_{2,2,\rho},
\end{aligned}$$

where we have successively used (18), the upperbound (9) of K_H, the fact that $\rho(s)\ge 1$ for any $s\in[0,1]$ and the Cauchy–Schwarz inequality. □

Remark 5.1. Note that when $H=1/2$, the contribution of the second order term A_2 is $1/2.\int_0^t \sigma^2(s)\ ds$ which is the term corresponding to the square bracket in the classical Itô formula. Still when $H=1/2$, $I_{0^+}^{H-1/2}=\mathrm{Id}$ and (29) differs from the Itô formula in [14] by only $1/2\int_0^t\sigma^2(s)\ ds$. Moreover when the processes σ and ξ and X_0 are $\{\mathscr{F}_t^H,\ t\in[0,1]\}$–adapted, all the terms involving Gross-Sobolev derivatives vanish when $H=1/2$ but not when $H>1/2$ because of the derivative of the kernel $K_H(t,.)$.

Corollary 5.1. *Let $H>1/2$ and $f:\mathbf{R}\to\mathbf{R}$ be a twice differentiable function with bounded derivatives. Define $u(t,x)=\mathbf{E}_H\left[f(x+W_t)\right]$, we have :*

$$\frac{\partial}{\partial t}u(t,x)=HV_H\ t^{2H-1}\ \frac{\partial^2}{\partial x^2}u(t,x).$$

Proof. Applying (31) with $\psi=1$, we obtain :

$$\partial_t u(t,x)=\mathbf{E}_H\left[F''(X_t)\right]t^{H-1/2}I_{0^+}^{H-1/2}(u^{1/2-H}K_H(t,u))(t).$$

Now, using Theorem [2.1], we see that for any $f\in L^2([0,1])$,

$$I_{0^+}^{H-1/2}(u^{1/2-H}f)(s)=s^{1/2-H}I_{0^+}^{-1}(K_Hf)(s).$$

Hence,

$$I_{0^+}^{H-1/2}\Big(u^{1/2-H}K_H(t,u)\Big)(t)=t^{1/2-H}\ \frac{\partial R_H}{\partial_s}(t,s)\Big|_{s=t}=HV_Ht^{H-1/2}$$

which proves the result. □

Before we state the Itô formula for processes defined by the stochastic integral of second kind, we need the following result :

Lemma 5.3. *For any $f \in L^2_\rho([0,1])$, we have*

$$\int_0^1 (\mathcal{K}_H f)(s)^2\, ds \le c \int_0^1 f(s)^2 s^{1/2-H}\, ds,$$

$$\int_0^1 (\mathcal{K}_H^* f)(s)^2\, ds \le c \int_0^1 f(s)^2 s^{1/2-H}\, ds,$$

$$\int_0^1 (\mathcal{K}_H \mathcal{K}_H^* f)(s)^2\, ds \le c \int_0^1 f(s)^2 s^{1-2H}\, ds,$$

where $\mathcal{K}_H = K_{1/2}^{-1} K_H$.

Proof. From Theorem (2.1), we know that

$$K_H^* f = x^{1/2-H} I_{1^-}^{H-1/2}(x^{H-1/2} f).$$

Since $I_{1^-}^{H-1/2}$ is a positive operator, using Cauchy–Schwarz inequality,

$$\begin{aligned}\int_0^1 (\mathcal{K}_H^* f)(s)^2\, ds &= \int_0^1 s^{1-2H} I_{1^-}^{H-1/2}(u^{H-1/2} f)(s)^2\, ds \\ &\le c \int_0^1 s^{1-2H} I_{1^-}^{H-1/2}(f)(s)^2\, ds \\ &\le c \int_0^1 s^{1-2H} (1-s)^{H-1/2} \int_s^1 (u-s)^{H-3/2} f(u)^2\, du\, ds \\ &\le c \int_0^1 f(u)^2 \int_0^u s^{1-2H} (u-s)^{H-3/2}\, ds\, du \\ &\le c \int_0^1 f(s)^2 s^{1/2-H}\, ds,\end{aligned}$$

where c denotes any constant. The other inequalities are shown similarly. □

Theorem 5.2. *Assume that for any $t \in [0,1]$,*

$$X_t = x_0 + \int_0^t \xi_s\, ds + \int_0^t \sigma_s\, \mathring{d}W_s,$$

where ξ belongs to $\mathbb{D}^H_{2,1,\rho}(L^2([0,1]))$ and $\sigma \in \mathbb{D}^H_{2,2,\rho}(L^2_\rho([0,1]))$. For any F twice differentiable with bounded derivatives, we have for any t, $\mathbf{P}_H$–almost surely :

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\xi_s\, ds + \int_0^t F'(X_s)\sigma_s\, \mathring{d}W_s + \int_0^t F''(X_s)\dot{D}_s X_s \sigma_s\, ds, \quad (33)$$

with

$$\dot{D}_t X_t = \int_0^t \dot{D}_t \xi_s \, ds + (\mathcal{K}_H \mathcal{K}_H^* \sigma)(t) + \int_0^t \dot{D}_t \sigma_s \, \overset{\circ}{d} W_s,$$

and $\dot{D}_t \phi$ *is defined as* $\mathcal{K}_H(\dot{\nabla}_. \phi)(t)$ *(see* (24)*).*

Sketch of the proof. Formally, the proof follows the lines of the preceding one. The third step consists also of upper–bounding the expectation of the right–hand–side of (33) by a polynomial in $\|F'\|_\infty$, $\|F''\|_\infty$, $\|\xi\|_{\mathbb{D}^H_{2,1,\rho}(L^2_\rho([0,1]))}$ and $\|\sigma\|_{\mathbb{D}^H_{2,2,\rho}(L^2_\rho([0,1]))}$. For instance,

$$\begin{aligned}
\mathbf{E}_H \left[| \int_0^t F'(X_s) \sigma_s \, \overset{\circ}{d} W_s | \right]^2 &\le \mathbf{E}_H \left[|\delta_H K_H \mathcal{K}_H^* (F'(X_.)\sigma)|^2 \right] \\
&\le \int_0^1 \mathcal{K}_H^*(F'(X_.)\sigma)(s)^2 \, ds \\
&\quad + \operatorname{trace}\Big(\nabla K_H \mathcal{K}_H^*(F'(X_.)\sigma) \circ \nabla K_H \mathcal{K}_H^*(F'(X_.)\sigma) \Big) \\
&\le \|F'\|_\infty^2 \|\mathcal{K}_H^* \sigma\|_{L^2}^2 + \|\nabla K_H \mathcal{K}_H^*(F'(X_.)\sigma)\|_{HS}^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\dot{\nabla}_r \mathcal{K}_H^*(F'(X_.)\sigma)(u) = \mathcal{K}_H^*(F''(X_.)\sigma \dot{n}_r X_.)(s) \\
+ \mathcal{K}_H^*(F'(X_.)\dot{\nabla}_r \sigma(.))(s).
\end{aligned}$$

Hence by Lemma [5.3], we have

$$\begin{aligned}
\mathbf{E}_H \left[| \int_0^t F'(X_s) \sigma_s \, \overset{\circ}{d} W_s | \right]^2 &\le c \|F'\|_\infty^2 \|\sigma\|^2_{\mathbb{D}^H_{2,1,\rho}(L^2_\rho([0,1]))} \\
&\quad + c\|F''\|_\infty^2 \|\sigma\|^2_{L^2_\rho} \Big(\|\sigma\|^2_{\mathbb{D}^H_{2,2,\rho}(L^2_\rho([0,1]))} + \|\xi\|^2_{\mathbb{D}^H_{2,1,\rho}(L^2([0,1]))} \Big),
\end{aligned}$$

where c is a constant. The other terms are treated similarly. □

Theorem 5.3. *Assume that for any* $t \in [0,1]$*,*

$$X_t = x_0 + \int_0^t \xi_s \, ds + \int_0^t \sigma_s \, \tilde{d} W_s,$$

where ξ *belongs to* $\mathbb{D}^H_{2,1,\rho}(L^2([0,1]))$ *and* $\sigma \in \mathbb{D}_{2,2,\rho}(L^2_\rho([0,1]))$*.*

1. *The integral* $\int_0^1 \dot{D}_t X_t \, dt$ *is finite, and*

2. *For any F twice differentiable with bounded derivatives, we have for any t, $\mathbf{P}_H$–almost surely :*

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\xi_s\, ds + \int_0^t F'(X_s)\sigma_s\, \tilde{d}W_s - \int_0^t F''(X_s)\dot{D}_s X_s \sigma_s\, \tilde{d}W_s.$$

Proof. By the computations we made in the previous proof, we have

$$\int_0^1 (\dot{D}_t X_t)^2\, dt \text{ is finite.}$$

The last part follows from the previous Itô formula. □

References

[1] M.A. Arcones. On the law of the iterated logarithm for gaussian processes. *Journal of Theoretical Probability*, 8(4):877–904, October 1995.

[2] R. J. Barton and H. Vincent Poor. Signal detection in fractional gaussian noise. *IEEE trans. on information theory*, 34(5):943–959, September 1988.

[3] J. Bertoin. Sur une intégrale pour les processus à α variation bornée. *The Annals of Probability*, 17(4):1521–1535, 1989.

[4] N. Bouleau and F. Hirsch. *Dirichlet Forms and Analysis on Wiener Space.* De Gruyter Studies in Mathematics, 1992.

[5] R. Coelho and L. Decreusefond. Video correlated traffic models for broadband communications networks. In *Proceedings ot the ITC Seminar on Teletraffic management*, 1995.

[6] D. Feyel and A. de La Pradelle. Fractional integrals and Brownian processes. Preprint, Université d'Évry, 1996, 1996.

[7] H. Föllmer. Calcul d'itô sans probabibilité. In *Séminaire de probabilités*, pages 143–150. Springer-Verlag, 1980.

[8] W.E Leland, M.S. Taqqu, W. Willinger, and D.V. Wilson. On the self-similar nature of ethernet traffic. *IEEE/ACM trans. Networking*, 2(1):1–15, Februar 1994.

[9] S.J. Lin. Stochastic analysis of fractional Brownian motions. To appear in Stochastics, 1996, 1996.

[10] B. Mandelbrot and J. Van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Review*, 10(4):422–437, October 1968.

[11] A.F. Nikiforov and V.B. Uvarov. *Special Functions of Mathematical Physics*. Birkhäuser, 1988.

[12] I. Norros. On the use of the fractional brownian motion in the theory of connectionless networks. *IEEE Jour. on Sel. Ar. in Communications*, 13(6):953–962, August 1995.

[13] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives*. Gordon & Breach Science, 1993.

[14] A. S. Üstünel. The Itô formula for anticipative processes with nonmonotonous time scale via the Malliavin calculus. *Probability Theory and related fields*, 79:249–269, 1988.

[15] A.S. Üstünel. *An Introduction to Analysis on Wiener Space*, volume 1610 of *Lectures Notes in Mathematics*. Springer-Verlag, 1995.

[16] A.S. Üstünel and M. Zakai. The constructions of filtrations on abstract wiener spaces. *Journal of Functional Analysis*, To appear 1996.

[17] S. Watanabe. Lectures on stochastic differential equations and Malliavin calculus. Tata Institute of Fundamental Research, 1984.

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