STOCHASTIC INTEGRATION WITH RESPECT TO FRACTIONAL BROWNIAN MOTION

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1. Introduction

In the past few years, the fractional Brownian motion has been the subject of numerous investigations. Its potential applications to telecommunications and mathematical finance are the practical reasons for which this process is so much studied. On the theoretical point of view, it is an interesting process because it is neither a Markov process nor a semi-martingale so that stochastic calculus with respect to it is challenging. In particular, several attempts have been made to define a *good* stochastic integral with respect to the fBm. This paper aims to describe the main approaches developed up to now and to show how they relate one to each other. This work is organized as follows: in Section 3, we give and compare the different definitions, existing in the current literature, of what could be a stochastic integral with respect to fBm. Section 4 is devoted to the usage we can make of these integrals to develop a stochastic calculus 'à la Itô', namely we focus on the basic tools such as Girsanov theorem, Clarke representation formula and Itô formula. At last, in Section 5, we illustrate how these tools can be applied to two concrete problems: the parametric estimation of the drift of a 'fractional diffusion' and the non-linear filtering problem. Notations and concepts of deterministic fractional calculus and Malliavin calculus are given in the appendices.

2. Preliminaries

Definition 2.1. *For any H in $(0,1)$, the fractional Brownian motion of index (Hurst parameter) H, $\{B_H(t);\ t \in [0,1]\}$ is the centered Gaussian process whose covariance kernel is given by*

$$R_H(s,t) = E\left[B_H(s)B_H(t)\right] \overset{def}{=} \frac{V_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

where

$$V_H \overset{def}{=} \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

Its main useful properties are (see [9, 11] and references therein for their proofs) :

i – The sample-paths of B_H are almost surely Hölder continuous of any order less than H.

ii – Its $1/H$-variation on $[0,t]$ is finite :

$$\lim_{n\to+\infty}\sum_{j=0}^{2^n-1}|B_H((j+1)2^{-n}\wedge t)-B_H(j2^{-n}\wedge t)|^p=\begin{cases}0 & \text{if } pH>1,\\ \infty & \text{if } pH<1,\\ V_H.t & \text{if } pH=1.\end{cases}$$

In particular, B_H is a Dirichlet process (i.e., its quadratic variation is null) for $H>1/2$ and has an infinite quadratic variation for $H<1/2$.

iii – There exists a standard Brownian motion $\{B(s),\, s\in[0,1]\}$ such that

$$B_H(t)=\int_0^t K_H(t,r)\,dB(r),\ \text{where} \tag{1}$$

$$K_H(t,r)=\frac{(t-r)_+^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}F(\frac{1}{2}-H,H-\frac{1}{2},H+\frac{1}{2},1-\frac{t}{r}). \tag{2}$$

The Gauss hyper-geometric function $F(\alpha,\beta,\gamma,z)$ (see [21]) is the analytic continuation on $\mathbb{C}\times\mathbb{C}\times\mathbb{C}\backslash\{-1,-2,\dots\}\times\{z\in\mathbb{C}, Arg|1-z|<\pi\}$ of the power series

$$\sum_{k=0}^{+\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}z^k.$$

Here $(\alpha)_k$ denotes the Pochhammer symbol defined by

$$(\alpha)_0=1 \text{ and } (\alpha)_k\overset{def}{=}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}=\alpha(\alpha+1)\dots(\alpha+k-1).$$

More important than the expression of $K_H(t,s)$ is the relation :

$$\int_0^{t\wedge r}K_H(t,s)K_H(r,s)\,ds=R_H(t,r). \tag{3}$$

This means that the linear integral operator of kernel K_H is a quasi-nilpotent square root of the integral operator of kernel R_H. Hence such a procedure can be done for any Gaussian process since such a square-root always exists – see [8].

We fix once for all a value of $H\neq 1/2$ so that, otherwise stated, we omit the index H everywhere for the sake of notational simplicity.

3. Constructions

The different definitions of stochastic integrals with respect to B_H can be sorted in two main groups : those which are relying on the sample-path properties of B_H and those which are based on its Gaussiannity.

3.1. **Sample-path defined integrals.** The very first idea which comes to mind when we try to construct a stochastic integral with respect to B_H is to consider the so-called Riemann sums :

$$RS_n(u)\overset{def}{=}\sum_{i=0}^{2^n-1}u_{i2^{-n}}(B_H((i+1)2^{-n})-B_H(i2^{-n})) \tag{4}$$

and then to identify the conditions on u which are sufficient to ensure the convergence (at least in probability) of this quantity.

Since for $H > 1/2$, the fBm is a Dirichlet process , (4) can be given a sense using the approach developed by Föllmer in [14]. On the other hand, since the fBm has $1/H$ bounded variation, one can use the work of Bertoin [3] in which it is proved that $RS_n(u)$ converges whenever u has $1/\beta$-bounded variation with $\beta + H > 1$ and $\beta \geq 2$. In the same vein, one can also cite the papers [7, 19] which consider more specifically the case of the fractional Brownian motion. Using the fact that for f and g two continuously differentiable functions, we have

$$|\int f dg| \leq c\|f\|_{\mathrm{Hol}(\alpha)}\|g\|_{\mathrm{Hol}(\beta)}$$

(see Appendix A for the definitions of the notations such as $\mathrm{Hol}(\alpha)$) where $\alpha + \beta \geq 1$, it is proved in [13] that $RS_n(u)$ converges provided that $u \in \mathrm{Hol}_+(1 - H)$ – for similar approaches, see [5]. Afterwards, we set

$$(\mathrm{RS})\text{-}\int u(s)\, dB_H(s) = \lim_{n\to\infty} RS_n(u),$$

whichever hypothesis are chosen to ensure its convergence. A slightly different approach appeared in [32].

Definition 3.1. *If u belongs to $\mathcal{B}_{\alpha,1}$ for some $\alpha > 1-H$, a stochastic integral of u with respect to B_H can be defined according to the fractional integration by parts formula :*

$$(5)\quad (\mathrm{FIP})\text{-}\int u(s)\, dB_H(s) = (-1)^\alpha \int_0^1 I_{0^+}^{-\alpha}(u_{0^+})(s) I_{1^-}^{\alpha-1}(B_H^{1^-})(s)\, ds + u(0^+)B_H(1^-),$$

where $u_{0^+}(s) = u(s) - u(0^+)$ and $u(0^+) = \lim_{\epsilon\downarrow 0} u(\epsilon)$. Similarly, $B_H^{1^-}(s) = B_H(1^-) - B_H(s)$ and $B_H(1^-) = \lim_{\epsilon\downarrow 0} B_H(1 - \epsilon)$.

In particular, if $u \in \mathrm{Hol}_+(1 - H)$, the integral $(\mathrm{FIP})\text{-}\int u(s)\, dB_H(s)$ is well defined and the process $t \mapsto (\mathrm{FIP})\text{-}\int u(s)\mathbf{1}_{[0,t]}(s)\, dB_H(s)$ belongs to $\mathrm{Hol}_-(H)$. Hence even though this definition is valid for any value of H in $(0, 1)$, one can iterate it (i.e., consider the stochastic integral of a process defined itself as a stochastic integral) only for $H > 1/2$, the value for which $H > 1 - H$. Moreover, in the case $H > 1/2$ we have :

Theorem 3.1 (cf. [32])**.** *If u is in $\mathrm{Hol}_+(1 - H)$, $(\mathrm{RS})\text{-}\int u(s)\, dB_H(s)$ exists and coincides with $(\mathrm{FIP})\text{-}\int u(s)\, dB_H(s)$.*

In all the definitions of this section, one did not care of the adaptedness of the integrand. Since all these definitions are made pathwise, it is only the sample-path regularity of the integrand that counts. It is thus not surprising that these stochastic integrals are not very suitable to a real stochastic calculus. Except for very specific case, it is in fact impossible to compute even the expectation of any of these integrals. More probabilistic approaches are given by the next methods.

3.2. **Wiener integrals.** Instead of using the sample-paths properties of B_H, we focuses now on its Gaussiannity. Stochastic integrals of deterministic functions with respect to a Gaussian process are well known since the early fifties (see [20] an references therein) and are called Wiener integrals. Their appealing property is that in the case of the usual Brownian motion, this integral coincides, for deterministic integrands, with the celebrated Itô integral. As we shall show below, this is no longer the case for other Gaussian processes. Nevertheless, there is still a strong relationship between the different kinds of approaches.

As a preliminary to the definition of Wiener integrals, we first fix a few notations and definitions: We denote by Ω the canonical space which is here the space of continuous functions from $[0,1]$ into $\mathbf{R}$, vanishing at time 0, equipped with its strong topology :

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|.$$

The probability measure P on Ω, is such that the canonical process, $\{B_H(\omega,t) = \omega(t),\ t \in [0,1]\}$, is a fractional Brownian motion of Hurst index H. As for any Gaussian processes, the so-called Reproducing Kernel Hilbert Space (RKHS for short), denoted by $\mathcal{H}$, is of paramount importance. It is defined as the closure of the vector space spanned by the set of functions $\{R(t,.),\ t\in[0,1]\}$ equipped with the scalar product :

$$< R(t,.), R(s,.) >= R(t,s) \text{ for all } t \text{ and } s.$$

Unfortunately, this general definition of the RKHS is clearly not a satisfying one. Actually, according to this definition, deciding whether a given function belongs to $\mathcal{H}$, is almost impossible. Meanwhile, it is well known that for the standard Brownian motion, $\mathcal{H}$ is nothing but the space of absolutely continuous functions, vanishing at 0, whose derivative belongs to $L^2([0,1])$. In the case of the fBm, it has been proved (in [9] for any H after a result of [2] for the case $H > 1/2$) that :

Theorem 3.2. *For any $H \in (0,1)$, $\mathcal{H}$ is the set of functions f which can be written as*

$$f(t) = \int_0^t K(t,s)\tilde{f}(s)ds,$$

with $\tilde{f}$ belonging to $L^2([0,1])$. By definition, $\|f\|_{\mathcal{H}} = \|\tilde{f}\|_{L^2}$.

It turns out that as a vector space (without any consideration about norms), $\mathcal{H}$ coincides with the space $I_{0^+}^{H+1/2}(L^2)$ (see appendix A for the definition of this space), which is the space of functions which are $(H + 1/2)$ differentiable, vanishing at 0, whose $(H + 1/2)$-th derivative belongs to L^2. This way, one can see the similarity between the standard case $(H = 1/2)$ and the case $(H \neq 1/2)$.

Definition 3.2. *As for any Gaussian process, the (abstract) Wiener integral with respect to the fractional Brownian motion is defined as the linear extension from $\mathcal{H}$ in $L^2(\Omega, P)$ of the isometric map :*

$$\mathcal{H} \longrightarrow L^2(\Omega, P)$$
$$R(t, .) \longmapsto B_H(t).$$

This means that :

$$\text{(Wiener)-}\int \sum_{i=1}^{n} \alpha_i R_H(t_i, s)\, dB_H(s) = \sum_{i=1}^{n} \alpha_i B_H(t_i).$$

For any (deterministic) function $u \in \mathcal{H}$, there exists a sequence $(u_n,\, n \geq 1)$, where each u_n is a finite linear combination of the $R(t, .)$, which converge to u in $\mathcal{H}$. By definition,

$$\text{(Wiener)-}\int u(s)\, dB_H(s) = L^2(\Omega, \mathrm{P}) - \lim_{n\to\infty} \text{(Wiener)-}\int u_n(s)\, dB_H(s).$$

If we apply this construction to the Brownian motion, because of the definition of the RKHS, we see that we can only integrate functions which are absolutely continuous and whose derivative is square integrable on $[0, 1]$. On the other hand, it is well known that it is sufficient to have the square integrability of the integrand to give a sense to a stochastic integral with respect to B. This difficulty is due to a simple and very often implicit identification. Actually, it does not change anything to the properties of Wiener integrals if we replace $\mathcal{H}$ by an isometrically isomorphic Hilbert space. Thus in the Brownian case, it is customary to identify $L^2([0, 1])$ and $\mathcal{H}$ since the map I^1_{0+} is a bijective isometry from the first space onto the other: if u belongs to L^2, then we identify u and $f(t) = \int_0^t u\, ds = (I^1_{0+}u)(t)$. It is only when the Wiener integral is defined this way that it is appears as the restriction of the Itô integral.

Let us now do the same sort of work for the fractional Brownian motion. By a representation of $\mathcal{H}$, we mean a pair composed of a functional space and a bijective isometry between this space and $\mathcal{H}$. By a concrete version of the Wiener integral with respect to the fBm, we mean a pair composed of a representation of $\mathcal{H}$ together with the family of functions $\{j(t),\, t \in [0, 1]\}$ which will be mapped to the W_t's in order to conserve the necessary isometry property, which is

$$\|j(t)\|_{\mathcal{H}} = \mathrm{E}\left[W_t^2\right].$$

As a trivial consequence of Theorem 3.2, we have :

Theorem 3.3. *There exists a canonical isometric bijection between $L^2([0, 1])$ and $\mathcal{H}$ given by :*

$$i_1 : L^2([0, 1]) \longrightarrow \mathcal{H}$$
$$h \longmapsto f(t) = \int_0^t K(t, s) h(s) ds.$$

Hence $(L^2([0,1]), i_1)$ *is a representation of* $\mathcal{H}$.

Note that when $H = 1/2$, i_1 is nothing but I^1_{0+}, the quadrature operator and that $i_1(\mathbf{1}_{[0,t]}) = K_{1/2}(t,.)$. More generally, according to (3), for any H, the predecessor of $R(t,.)$ by i_1 is $K(t,.)$.

Another representation, well defined only for $H > 1/2$, is given by the following theorem :

Theorem 3.4 ([2, 12, 22]). *For any* $H > 1/2$, *consider* $L^2([0,1])$ *equipped with the twisted scalar product :*

$$< f, g >= \iint_{[0,1]^2} f(s)g(t)|t-s|^{2H-2}\, ds\, dt.$$

First define the linear map i_2 *on step functions by :*

$$\begin{aligned} i_2 : (L^2([0,1]), <,>) &\longrightarrow \mathcal{H} \\ \mathbf{1}_{[0,t]} &\longmapsto R(t,.). \end{aligned}$$

Denote by i_2, *the extension of this map to the whole of* $\mathcal{H}^2 \overset{not}{=} closure(L^2([0,1]), <,>)$. *Then* $(\mathcal{H}^2, i_2)$ *is a representation of* $\mathcal{H}$.

Remark 3.1. Other characterizations of the RKHS of the fBm are given in [24]. In particular, it is proved there that $(L^2([0,1]), <,>)$ is not complete and that explains that we need to take its closure to obtain a Hilbert space.

To see the connection between this paper and [24], we may transform the expression of $K(t,s)$. After [29]): we know that for $f \in L^2$,

$$Kf = I^1_{0+} x^{H-1/2} I^{H-1/2}_{0+} x^{1/2-H} f. \tag{6}$$

We introduce K^*, the adjoint of K, i.e., for any f and g two square integrable functions, K^*g is defined by the condition:

$$\int_0^1 Kf(s)g(s)\, ds = \int_0^1 f(s)K^*g(s)\, ds.$$

By Fubini's Theorem, it turns out that

$$K^*g(t) = \int_t^1 K(s,t)g(s)\, ds$$

and at least formally, $K(t,s) = K^*(\epsilon_t)(s)$, where ϵ_t denotes the Dirac mass at time t. On the other hand, we deduce from (6) that

$$K^*g = x^{1/2-H} I^{H-1/2}_{1-} x^{H-1/2} I^1_{1-}. \tag{7}$$

Since $I^1_{0+}(\epsilon_t) = \mathbf{1}_{[0,t]}$, it follows that

$$K(t,s) = s^{1/2-H}\big(I^{H-1/2}_{1-} x^{H-1/2} \mathbf{1}_{[0,t]}\big)(s).$$

This is precisely the expression used in [24] to prove that for $H < 1/2$,

$$i_3 : L^2([0,1]) \longrightarrow \mathcal{H}$$
$$h \longmapsto f(t) = t^{1/2-H}\big(I_{1^-}^{H-1/2} x^{H-1/2} h\big)(t),$$

with $\|f\|_{\mathcal{H}} = \|h\|_{L^2}$, is a representation of $\mathcal{H}$. Actually, rephrased in the setting of this paper, this representation is the 'dual' representation of i_1. Remind that $\mathcal{H}$ is densely included in the space of continuous functions on $[0,1]$, hence its dual space contains the linear combination of Dirac masses as a dense subspace. Henceforth, in virtue of (7), i_3 is strictly speaking a representation of the dual space of $\mathcal{H}$, since this space is isometrically isomorphic to $\mathcal{H}$ and since we work up to isomorphisms, i_3 can also be viewed as a representation of $\mathcal{H}$.

Now, one can consider the Wiener integrals as the extensions of the following isometries :

$$(8) \quad \text{(W1)-}\int : L^2([0,1]) \longrightarrow L^2(\Omega, \mathrm{P}) \quad \text{or} \quad \text{(W2)-}\int : \mathcal{H}^2 \longrightarrow L^2(\Omega, \mathrm{P})$$
$$K(t,.) \longmapsto W_t \qquad\qquad \mathbf{1}_{[0,t]}(.) \longmapsto W_t.$$

These definitions require a few remarks :

- Note that mapping (W1)-$\int \mathbf{1}_{[0,t]}(s)\, dB_H(s)$ to $B_H(t)$ would be inconsistent with the isometry property required by the abstract scheme of the Wiener integral since

$$(9) \quad \mathrm{E}\left[|\text{(W1)-}\int \mathbf{1}_{[0,t]}(s)\, dB_H(s)|^2\right] = \|\mathbf{1}_{[0,t]}\|_{L^2}^2 = t \neq \mathrm{E}\left[|B_H(t)|^2\right] = V_H t^{2H}.$$

- As a consequence of the definition of a Wiener integral and of (9), the process (W1)-$\int \mathbf{1}_{[0,t]}(s)\, dB_H(s)$, $t \geq 0$ is a centered Gaussian process whose covariance kernel is $\min(t,s)$, hence it is a standard Brownian motion which we denote by $\tilde{B}$. Moreover, for the very same reasons, for any u deterministic belonging to L^2,

$$\text{(W1)-}\int u(s)\, dB_H(s) = \int u(s)\, d\tilde{B}_s,$$

  where the right-hand-side integral has to be understood in the Itô sense.
- So either we keep the original scalar product on $L^2([0,1])$ and we have to change the pre-image of $B_H(t)$ to $K(t,.)$ or we change the scalar-product on $L^2([0,1])$ so that we can keep $\mathbf{1}_{[0,t]}$ as the predecessor of $B_H(t)$. This is the main point where the situation for the fBm differs from the standard case. All the following difficulties come in fact from this Gordian knot.
- A consequence of changing the scalar-product appears in the computation of the expectation of $B_H(t)$ given the past $(B_H(u),\, u \leq r)$ for some $r < t$. In both cases, one has to express the orthogonality relations:

$$\mathrm{E}\left[B_H(u)(B_H(t) - \mathrm{E}\left[B_H(t) \,|\, B_H(u), u \leq r\right])\right] = 0,$$

in terms of orthogonality equations in $L^2([0,1])$, respectively $\mathcal{H}^2$. Explicit computations turn to be straightforward with (W1)-$\int$ and we easily obtain that :

Lemma 3.1. *For any* $0 \le r \le t \le 1$,

$$E[B_H(t) \,|\, B_H(u), u \le r] = \text{(W1)-}\int K(t,s)\mathbf{1}_{[0,r]}(s)\, dB_H(s).$$

The corresponding formula (for $H > 1/2$ only) in terms of (W2)-$\int$ can be deduced from the following identity (the *explicit* expression of the previous conditional expectation in terms of (W2)-$\int$ is given in [22].) :

Theorem 3.5. *For* $H > 1/2$, *for any function* $u \in L^2([0,1])$,

$$\text{(W1)-}\int K^* K^{*-1}_{1/2} u(s)\, dB_H(s) = \text{(W2)-}\int u(s)\, dB_H(s), \tag{10}$$

where K^* *denotes the adjoint of the integral operator of kernel* K, *i.e., for* f *and* g *in* $L^2([0,1])$,

$$\int_0^1 (Kf)(s)g(s)\, ds = \int_0^1 f(s)(K^*g)(s)\, ds.$$

Remark 3.2. We denote by $\mathcal{K}$ the map $K^{-1}_{1/2} \circ K$, since $K_{1/2}$ coincides with $I^1_{0^+}$, we get $\mathcal{K} = I^{-1}_{0^+}K$ that is to say

$$\mathcal{K}f(t) = (Kf)'(t).$$

For $H > 1/2$, according to (6), we get

$$\mathcal{K}u(t) = t^{H-1/2} I^{H-1/2}_{0^+}(x^{1/2-H}u)(t),$$

thus

$$\mathcal{K}^*u(t) = t^{1/2-H} I^{H-1/2}_{1^-}(x^{H-1/2}u)(t),$$

which according to the definition of fractional integrals turns out to be

$$\mathcal{K}^*u(t) = \frac{t^{1/2-H}}{\Gamma(H-1/2)} \int_t^1 (r-t)^{H-3/2} r^{H-1/2}\, u(r)\; dr.$$

For $H = 1/2$, $\mathcal{K}$ is the idendity map. For $H < 1/2$, $\mathcal{K}$ is a densely defined, closable operator from L^2 into itself which admits $I^{1/2-H}_{0^+}(L^2)$ as a core (i.e., as a dense subset of its domain) – see [10].

Proof of (10). By their very definition, K and $K_{1/2}$ are such that

$$K^*(\epsilon_t)(s) = K(t,s) \text{ and } K^*_{1/2}(\epsilon_t)(s) = I^1_{1^-}(\epsilon_t)(s) = \mathbf{1}_{[0,t]}(s),$$

where ϵ_t denotes the Dirac mass at time t. Hence $\mathcal{K}^*\mathbf{1}_{[0,t]} = K(t,.)$ and (10) is true for step functions of the form $\sum_{i=1}^n u_i \mathbf{1}_{[0,t_i]}$. Since $\mathcal{K}^*$ is a continuous map from $L^2([0,1])$ into itself, the general case follows by a trivial limiting procedure. □

- The consequence of changing the pre-image of W_t to something different from $\mathbf{1}_{[0,t]}$ appears in the Riemann-sums approach of the definition of a stochastic integral. When $H = 1/2$, $K_{1/2}(t,.) = \mathbf{1}_{[0,t]}$ so that we have :

$$\text{(W1)-}\int u(s)\, dB_H(s) = \text{(W2)-}\int u(s)\, dB_H(s) = \lim_{n\to+\infty} \sum_{i=0}^{2^n-1} u(i2^{-n})(B((i+1)2^{-n}) - B(i2^{-n})),$$

  for any continuous u. From (10), it is clear that (W1)-$\int$ and (W2)-$\int$ differ and from (8), it is clear that (W1)-$\int$ no longer coincides with (RS)-$\int$, whereas (W2)-$\int$ does coincide with (RS)-$\int$.

At this point, we thus have two difficulties: we have only used Wiener integrals and that means we can integrate only deterministic integrands. The second point is that we have two integrals at our disposal, namely (W1)-$\int$ and (W2)-$\int$ and none is strictly is better than the other. The latter may be more appealing because it coincides with the Riemann sums and thus has some reminiscences of Riemann-Stieltjes framework of integration. However, it would be a great mistake to neglect (W1)-$\int$ since as we'll see in section 5, it is the convenient tool to express and prove crucial theorems such as Girsanov Theorem and Clark Formula. In virtue of (10), it is always possible to rewrite a result expressed in terms of (W1)-$\int$ in terms of (W2)-$\int$ but the result is untractable.

We will see in the next section that the first difficulty (which is only of technical order) can be easily overcome whereas the second point is really unavoidable. Actually, for any Gaussian process distinct from the standard Brownian motion, the situation will be the same as it is for the fBm. It is the Brownian motion which is a singularity and the other processes which follow the common rules.

3.3. **Skohorod-like integrals.** The Malliavin calculus provides the convenient framework to extend the Wiener integral to random, even non-adapted, integrands. For the sake of simplicity, we won't go into the details of the Malliavin calculus but we will only summarize the useful results.

The basic tool for the following is in fact the Skohorod integral (see Appendix B) with respect to the standard Brownian motion B mentioned in property (iii) of section 2. The Skohorod integral of a process u is usually denoted by $\int u(s) \circ dB(s)$ and its definition can be found in several textbooks (see [30]) and is recalled at the end of this chapter. Two of its main properties are that :

- The expectation of $\int u(s) \circ dB(s)$ is null.
- For u adapted and square integrable, $\int u(s) \circ dB(s)$ coincides with the Itô integral of u. The space of processes which are Skohorod integrable, denoted by $\operatorname{Dom}\delta$ is much wider than the set of adapted and square integrable processes. For instance, any functional $f(B(t_1),\ldots,B(t_n))$ where

t_i belongs to $[0,1]$ for any $i \in \{1,\dots,n\}$ and f is a Lispchitz continuous function from $\mathbf{R}^n$ into $\mathbf{R}$ is Skohorod integrable but not Itô integrable. For all of this reasons, one can say that the Skohorod integral is an extension of the Itô integral (which itself is an extension of the Wiener integral $_{(\text{W1})}\text{-}\int$.

Moreover, it is proved in [11] that for deterministic processes, the Skohorod integral coincides with $_{(\text{W1})}\text{-}\int$. We thus have solved half of our problem by extending the first Wiener integral by the Skohorod integral, which we will denote from now on by $_{(\text{Sko1})}$-. To extend $_{(\text{W2})}\text{-}\int$, we simply use (10):

Definition 3.3. *For u such that $\mathcal{K}^* u$ belongs to* $\operatorname{Dom}\delta$, *we consider :*

$$_{(\text{Sko2})}\text{-}\int u(s) \circ dB_H(s) \stackrel{def}{=} \int \mathcal{K}^* u(s) \circ dB(s), \tag{11}$$

where the right-hand-side denotes the Skohorod integral of $\mathcal{K}^* u$.

Since K is lower-triangular (i.e., $K(t,s) = 0$ when $s > t$), we have:

$$\mathcal{K}^*_{t-} u \equiv \mathcal{K}^*_{1-}(u\mathbf{1}_{[0,t]}),$$

where $\mathcal{K}^*_{t-}$ denotes, by analogy with the notation I^{α}_{b-}, the adjoint of $\mathcal{K}$ in $L^2([0,t])$, i.e.,

$$\int_0^t \mathcal{K}^f(s) g(s)\, ds = \int_0^t f(s) \mathcal{K}^*_{t-} g(s)\, ds$$

for sufficiently regular f and g.

Henceforth, there is no ambiguity to set

$$_{(\text{Sko2})}\text{-}\int_0^t u(s) \circ dB_H(s) = \int \mathcal{K}^*_{t-} u(s) \circ dB(s).$$

Remark 3.3. This integral first appeared in [11] for $H > 1/2$ and coincides with the integral defined in [12] by the mean of Wick products.

Remark 3.4. It is also remarkable that $_{(\text{Sko2})}\text{-}\int$ is the integral which pops up when we try to define heuristically a stochastic integral with respect to B_H. Say we want to define something like $\int u(s)\dot{B}_H(s)\, ds$ where $\dot{B}_H$ aims to be the derivative of B_H – which rigourously speaking does not exist. Using relation (1) and the fact that the derivative operator is nothing but $K^{-1}_{1/2}$, we can say that :

$$\int u(s)\, \dot{B}_H(s)\, ds = \int u(s)\, K^{-1}_{1/2} \circ K(\dot{B})(s)\, ds.$$

By taking the adjoint map of $K^{-1}_{1/2} \circ K$, which is in fact $\mathcal{K}$, we obtain

$$\int u(s) \dot{B}_H(s)\, ds = \int \mathcal{K}^* u(s) \dot{B}(s)\, ds,$$

which is the empirical version of (11).

Remark 3.5. Note that $(\text{Sko2})\text{-}\!\int$ is an anticipative integral. For instance, for $H > 1/2$, since we have (see [29]) :

$$Kf = I^1_{0+} x^{H-1/2} I^{H-1/2}_{0+} x^{1/2-H} f, \tag{12}$$

the computation of $\mathcal{K}^* f(t)$ needs the knowledge of f between time t and 1 :

$$\mathcal{K}^* f(t) = \frac{t^{1/2-H}}{\Gamma(H-1/2)} \int_t^1 x^{H-1/2}(x-t)^{H-3/2} f(x)\, dx.$$

For $H < 1/2$, the problem is the same. However, note that for u adapted, $(\text{Sko2})\text{-}\!\int_0^t u(s) \circ dB_H(s)$ belongs to $\mathcal{F}_t$ even if $\mathcal{K}^*_{t-} u$ is an anticipative process.

The key point here is that by taking the adjoint map of $\mathcal{K}$, we proceed to the analog of an integration by parts and this explains how an anticipative integral appears. Namely, when one wants to compute $\int_0^1 wv\, dx$ with $v(x) = \int_0^x v'(y)\, dy$. We obtain by a classical integration by parts (including the trace terms in the integral) or by Fubini's theorem,

$$\int_0^1 wv\, dx = \int v'(x) \int_x^1 w(y)\, dy\, dx,$$

so that we actually have a sort of an "anticipative" integral of u.

Proposition 3.1 (see [11]). *For any H, provided that u belongs to the domain of $\mathcal{K}$ (which is true if u is suare integrable for $H > 1/2$ and which is satisfied whenever u belongs to $I^{1/2-H}_{0+} L^2$ for $H < 1/2$), the stochastic integral $(\text{Sko2})\text{-}\!\int$ coincides with the stochastic integral defined by Riemann sums for deterministic processes. We have the following identity provided that u is deterministic and both sides exist :*

$$(\text{Sko2})\text{-}\!\int u(s) \circ dB_H(s) = \lim_{|\pi_n| \to 0} \sum_{t_i \in \pi_n} u(t_i)(B_H(t_{i+1}) - B_H(t_i)).$$

When u is a "regular" random process, we have :

$$(\text{Sko2})\text{-}\!\int u(s) \circ dB_H(s) = \lim_{|\pi_n| \to 0} \sum_{t_i \in \pi_n} u(t_i)(B_H(t_{i+1}) - B_H(t_i)) - \int_0^1 D_s u(s)\, ds. \tag{13}$$

The term "regular" and the operator D are defined in appendix B, definition B.2.

Equation (13) leads naturally to define :

Definition 3.4. *For u regular, we define :*

$$(\text{Sko3})\text{-}\!\int u(s) \circ dB_H(s) = (\text{Sko2})\text{-}\!\int u(s) \circ dB_H(s) + \int_0^1 D_s u(s)\, ds$$

and

$$\text{(Sko3)-}\int_0^t u(s) \circ dB_H(s) = \text{(Sko2)-}\int_0^t u(s) \circ dB_H(s) + \int_0^t D_s u(s)\, ds.$$

The similarity of (Sko3)-$\int$ with a Stratonovitch-Skohorod type integral (see [23]) leads us to look at the limit of

$$LI_n(u) = \sum_{i=0}^{n-1} n \int_{i/n}^{(i+1)/n} u(s)\, ds\ (B_H((i+1)/n) - B_H(i/n)).$$

That is to say we look at the limit of the sum obtained by substituting the linear interpolation of the fBm to the differential element $dB_H(s)$.

Theorem 3.6 (see [10]). *For $H > 1/2$, for $u \in L^2(\Omega \times [0,1], dP \otimes dt) \cap \operatorname{Dom}\delta$ and such that ∇u belongs to $L^p(\Omega; L^p \otimes L^p)$, u is regular and the sequence $\{LI_n(u),\, n \geq 1\}$ converges in $L^2(\Omega, dP)$ to (Sko3)-$\int u(s) \circ dB_H(s)$.*

Theorem 3.7 (see [10]). *For $H < 1/2$, for $u \in \mathbb{D}_{2,1}(\mathcal{B}_{3/2-H,2})$, u is regular and the sequence $\{LI_n(u),\, n \geq 1\}$ converges in $L^2(\Omega, dP)$ to (Sko3)-$\int u(s) \circ dB_H(s)$.*

3.4. **Other constructions.** At least two other approaches are used to construct a stochastic integral with respect to B_H.

- In [1], B_H is expressed as the limit in probability of a sequence of semi-martingales. The stochastic integrals with respect to these semi-martingales being well-defined, the stochastic integral with respect to B_H is defined as their limit when it exists. Up to a slight change of notations, the expression of this integral coincides with that of (Sko3)-$\int$.
- Another kind of limiting procedure is used in [25]. First define the following two spaces :

  $$\mathcal{C}_0^\infty([0,1]^2) = \{k \in \mathcal{C}^\infty,\, k(1,r) = k(0,r) = 0, r \in [0,1]\},$$

  and let W be the completion of $\mathcal{C}_0^\infty([0,1]^2)$ under the norm :

  $$\|k\|_W^2 = \|k\|^2_{L^2([0,1]^2)} + \int_0^1 \partial_s \int_0^1 k^2(s,r)\, dr\, ds.$$

  A square integrable process is said to be W-integrable whenever for any sequence $(k_n,\, n \in \mathbb{N})$ whose elements belong to $\mathcal{C}_0^\infty([0,1]^2)$, converging to k in W, we have $\int_0^1 u(s)\partial_s k_n(s,.)\, ds \in \operatorname{Dom}\delta$ for any n and the limit of

  $$W_n^k(u) = \int_0^1 \Big(\int_0^1 u(s) \partial_s k_n(s,.)\, ds \Big) \circ dB(s)$$

  exists in $L^2(\Omega, \mathrm{dP})$ and is independent of the choice of $(k_n,\, n \geq \mathbb{N})$. Actually, considering $\int_0^1 u(s)\partial_s k_n(s,.)\, ds$ is a way to approach $\mathcal{K}^* u$ (by regularizing K so that Kf becomes differentiable). Thus when a process is W-integrable and its (Sko2)-$\int$ integral is well defined, the latter coincides with the limit of $W_n^k(u)$.

4. Stochastic calculus

One step is to construct a stochatic integral, the other is to be able to state some theorems using it. The situation for $H \neq 1/2$, will be again very different from the standard one in the sense that we will have to choose one construction or the other according to the theorem we want to state. For the Girsanov theorem, (Sko1)-$\int$ is the most adequate tool :

Theorem 4.1 (Girsanov theorem, see [11]). *Let u be a square integrable adapted process such that*

$$E[\Lambda_1^u] = 1 \text{ where } \Lambda_t^u = \exp(\text{(Sko1)-}\int_0^t u(s)\, dB_H(s) - \frac{1}{2}\int_0^t u(s)^2\, ds).$$

Let P_u be the probability defined by

$$\left.\frac{dP_u}{dP}\right|_{\mathcal{F}_t^H} = \Lambda_t^u.$$

The law of the process $\{B_H(t) - \int_0^t K(t,s)\, ds,\ t \in [0,1]\}$ under P_u is the same as the law of the process B_H under P. More generally, for any v square integrable and adapted process, the law of the process $\{\int_0^t K(t,s)v(s)\, dB(s) - \int_0^t K(t,s)u(s)v(s)\, ds,\ t \in [0,1]\}$ under P_u is the same as the law of the process $\{\int_0^t K(t,s)v(s)\, dB(s),\ t \in [0,1]\}$ under P.

Remark 4.1. For non-adapted shifts, Girsanov theorem still holds under convenient hypothesis (see [31]) but it is always expressed in terms of (Sko1)-$\int$.

According to the relation (10), this theorem can be expressed in terms of (Sko2)-$\int$ but the Radon-Nykodym derivative is much less easily computed – see [22] for such an approach. The point is that since B_H and B generates the same filtration, every $(\Omega; \mathcal{F}, \mathrm{P})$ martingale is in fact a martingale with respect to the standard Brownian motion B and hence can be expressed as a Itô stochastic integral with respect to B.

For the very same reason, the Itô-Clark is easily expressed as :

Theorem 4.2 (Itô–Clark representation formula). *For any square integrable random variable F, there exists a Skohorod integrable process such that*

$$F - E[F] = \int_0^1 u(s) \circ dB(s)$$

If F belongs to $\mathbb{D}_{2,1}$, then u can be computed by :

$$u(s) = E[\nabla_s F \,|\, \mathcal{F}_s].$$

The Itô formula is usually one of the most useful tools in stochastic calculus. Several proofs of a Itô formula for processes defined as stochastic integrals with respect to B_H do exist but they do not seem to be as useful as the classical formula (see section 5.2).

Assume in a first part that H is greater than 1/2. In this case, the fractional Brownian motion has a zero quadratic variation hence it is a Dirichlet process. It is well known that for such processes there exists an Itô formula (see [14, 27, 28, 32]) of the form :

$$F(B_H(t)) = F(B_H(0)) + \text{(RS)-}\int F'(B_H(s))\mathbf{1}_{[0,t]}(s)\,dB_H(s), \tag{14}$$

where F is of class $\mathcal{C}^1$. In fact, in our situation, the results known for Dirichlet processes can be made more explicit. One of the most important properties of semi-martingales is that a semi-martingale under a probability P is still a semi-martingale under any probability absolutely continuous with respect to P. Theorem 4.1 stands that processes of the form

$$X_t = x_0 + \int_0^t K(t,s)\xi(s)\,ds + \int_0^t K(t,s)\sigma(s)\,\circ dB(s), \tag{15}$$

satisfy the same property so they coudl be taken here as the analog of semi-martingales for the fBm.

Note that for σ constant, the second term reduces to B_H. One could had a "classical drift term" in the form $\int_0^t \alpha(s)\,ds$, but technically speaking, it behaves "classically" in the sense that in the Itô formula, there is no cross term between it and the terms involving the Brownian motion and moreover, in view of the Girsanov, this form of drift is non logical.

Let $\rho(s) = s^{1-2H}$ and denote set

$$L^2_\rho([0,1]) = \{h : [0,1] \to \mathbf{R},\ \int_0^1 h^2(s)\rho(s)\,ds < +\infty\}.$$

Note that when $H > 1/2$, $L^2_\rho([0,1]) \subset L^2([0,1])$ because $\rho(s) \geq 1$ for any $s \in [0,1]$. The weighted Sobolev space $\mathbb{D}_{2,k,\rho}$ is the set of elements ψ of $\mathbb{D}_{2,k}$ such that for any $n \leq k$, the n-th Gross-Sobolev derivative of ψ belongs to $L^2_\rho([0,1])^{\otimes n}$ and the norm of ψ in this space is defined by

$$\|\psi\|^2_{2,k,H,\rho} = \|\psi\|^2_{L^2(\mathrm{P})} + \sum_{i=1}^k \mathrm{E}\left[\|\nabla^{(i)}\psi\|^2_{L^2_\rho([0,1])^{\otimes i}}\right].$$

Theorem 4.3 (cf [11])**.** *Suppose $H > 1/2$, let $F : \mathbf{R} \longmapsto \mathbf{R}$ be twice differentiable with bounded derivatives and X be a process of the form given by*

(15), where ξ is in $\mathbb{D}_{2,1,\rho}(L^2_\rho([0,1]))$ and σ is in $\mathbb{D}_{2,2,\rho}(L^2_\rho([0,1]))$. We have

$$(16)\quad F(X_t)-F(X_0)=\int_0^t F'(X_s)(\mathcal{K}\xi)(s)\;ds$$
$$+\int_0^t \sigma(s)\mathcal{K}^*_{t^-}(F''\circ X\,K_H(\nabla_s\xi))(s)\;ds$$
$$+\int_0^t \sigma(s)\mathcal{K}^*_{t^-}(F'\circ X)(s)\;\circ dB(s)$$
$$+\int_0^t \mathcal{K}^*_{t^-}(F''\circ XK(.,s))(s)\,\sigma(s)^2\;ds$$
$$+\int_0^t \mathcal{K}^*_{t^-}(F''\circ X)\int_0^1 \nabla_s\sigma(r)K_H(.,r)\circ dB(r))(s)\sigma(s)\quad ds.$$

Remark 4.2. As a consequence, it is apparent that the family of processes of the form (15) is not stable by a non-linear transformation. Actually, it is an open question to find such a family of processes stable by non-linear transformations.

One can also wonder what looks like the non-linear transformation of a Skohorod integral. The answer is given by the following formula.

Theorem 4.4 (cf [11]). *Assume that for any $t\in[0,1]$,*

$$Z_t=z_0+\int_0^t \xi(s)\,ds+{\scriptstyle(\mathrm{Sko2})\text{-}}\!\int \sigma(s)\;\circ dB_H(s),$$

where ξ belongs to $\mathbb{D}_{2,1,\rho}(L^2([0,1]))$ and $\sigma\in\mathbb{D}_{2,2,\rho}(L^2_\rho([0,1]))$. For any F twice differentiable with bounded derivatives, we have for any t, P–almost surely :

$$(17)\quad F(Z_t)=F(z_0)+\int_0^t F'(Z_s)\xi(s)\,ds+{\scriptstyle(\mathrm{Sko2})\text{-}}\!\int_0^t F'(Z_s)\sigma(s)\;\circ dB_H(s)$$
$$+\int_0^t F''(Z_s)D_sZ_s\sigma(s)\,ds.$$

This can nicely expressed in terms of ${\scriptstyle(\mathrm{Sko3})\text{-}}\!\int$ by

$$F(Z_t)=F(z_0)+\int_0^t F'(Z_s)\xi(s)\,ds+{\scriptstyle(\mathrm{Sko3})\text{-}}\!\int_0^t F'(Z_s)\sigma(s)\;\circ dB_H(s).$$

Remark 4.3. For $X(t)=B_H(t)$, formula (16)as well as formula (17) read as

$$F(B_H(t))=F(0)+{\scriptstyle(\mathrm{Sko2})\text{-}}\!\int_0^t F'(B_H(s))\circ dB_H(s)$$
$$+HV_H\int_0^t F''(B_H(s))s^{2H-1}\,ds.$$

For $0<H<1/2$, there exists a formula only for non-linear transformation of B_H itself (see [4, 25, 26]) and it involves several other terms.

5. Applications

We now give two applications of the preceding theorems.

5.1. An estimation problem. Consider the process X defined by

$$X_t = \theta t + B_H(t)$$

and let P_X denote its law. We aim to estimate θ through the observation of a sample-path of X over $[0, t]$. Let ϕ be such that $(K\phi)(t) = t$, by the Girsanov Theorem

$$\left.\frac{d\mathrm{P}_X}{d\mathrm{P}}\right|_{\mathcal{F}_t} = \exp(\theta\,(\mathrm{W1})\text{-}\int_0^t \phi(s)\,dB_H(s) - \frac{\theta^2}{2}\int_0^t \phi(s)^2\,ds) \text{ or equivalently,}$$

$$\left.\frac{d\mathrm{P}}{d\mathrm{P}_X}\right|_{\mathcal{F}_t} = \exp(-\theta a\,(\mathrm{W1})\text{-}\int_0^t \phi(s)\,dB_H(s) + \frac{\theta^2}{2}\int_0^t \phi(s)^2\,ds), \text{ i.e.,}$$

$$\mathrm{E}\,[F] = \mathrm{E}\left[F(X(w))\exp(-\theta a\,(\mathrm{W1})\text{-}\int_0^t \phi(s)\,dB_H(s) + \frac{\theta^2}{2}\int_0^t \phi(s)^2\,ds)\right].$$

Hence the P maximum likelihood ratio estimate $\hat{\theta}_t$ of θ is

$$\hat{\theta}_t = \frac{1}{\int_0^t \phi(s)^2\,ds}(\mathrm{W1})\text{-}\int_0^t \phi(s)\,dB_H(s) = \frac{1}{\int_0^t \phi(s)^2\,ds}(\mathrm{W2})\text{-}\int_0^t \mathcal{K}^{*-1}\phi(s)\,dB_H(s).$$

The functions ϕ and $\mathcal{K}^{*-1}\phi$ can be computed, using (12), by searching a monomial solution and we find:

$$\phi(s) = \frac{\Gamma(3/2 - H)}{\Gamma(2 - 2H)} s^{1/2 - H}$$

$$\text{and } \mathcal{K}^{*-1}\phi(s) = \frac{\Gamma(3/2 - H)}{\Gamma(2 - 2H)\Gamma(H + 1/2)}\left(\frac{s}{1 - s}\right)^{1/2 - H}.$$

5.2. Non-linear filtering. This application is here to show that the Itô formula is not as useful as usual but that an infinite dimensional approach may be more fruitful.

Assume that we are given the observation Y of a signal X corrupted by a fractional Brownian motion noise. Assume furthermore that the original signal X, satisfies an evolution equation involving an other fBm with a possibly different Hurst index. The filtering problem consists of finding the conditional law of X with respect to Y. The problem in its whole generality has been addressed in [6] – some simpler filtering problems have also been studied in [15, 16, 18]. The usual strategy can be replicated here with some additional technical difficulties due to the lack of martingale properties. In order to show the main ideas, we will explicitly handle the following simple case :

$$X_t = B_{H_1}(t)$$

$$Y_t = \int_0^t K_{H_2}(t, s) h(X_s)\,ds + B_{H_2}(t),$$

where B_{H_1} and B_{H_2} are two independent fractional Brownian motion of respective Hurst index H_1 and H_2.

Theorem 5.1. *Let Q be the probability measure defined by*

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} \overset{def}{=} \Lambda_t^h = \exp\Big(\text{Sko1} - \int_0^t h(X_s)dB_{H_2}(s) - \frac{1}{2}\int_0^t h(X_s)^2\, ds)$$

Under Q, the processes :

$$\tilde{B}_{H_1}(t) = B_{H_1}(t) \quad \textit{and}\ \tilde{B}_{H_2}(t) = B_{H_2}(t) - \int_0^t K_{H_2}(t,s)h(X_s)ds,$$

are two independent fractional Brownian motion of Hurst index H_1 and H_2.

We give the proof to illustrate the role of the infinite dimension.

Proof. Let B^i be the standard Brownian motion

$$B^i(t) = {\scriptstyle (\text{W1})\text{-}}\int \mathbf{1}_{[0,t]}(s)\, dB_H(s).$$

For any $(t_1,\dots,t_n) \in [0,1]^n$, consider the $2n$-dimensional P martingale

$$Z_r^{t_1,\dots,t_n} = (\int_0^r K_{H_1}(t_j,s)\, dB^1(s),\ \int_0^r K_{H_2}(t_j,s)\, dB^2(s))_{j=1,\dots,n}$$

and the bounded variation process

$$A_r^{t_1,\dots,t_n} = (0, \int_0^r K_{H_2}(t_j,s)\, h(X_s)\, ds)_{j=1,\dots,n}.$$

The classical Girsanov theorem states that for any bounded f,

$$\mathrm{E}_{\mathrm{Q}}\left[f(Z_r^{t_1,\dots,t_n} - A_r^{t_1,\dots,t_n})\right] = \mathrm{E}\left[f(Z_r^{t_1,\dots,t_n}))\right].$$

Taking $r = \sup_i t_i$ yields to

$$\mathrm{E}_{\mathrm{Q}}\left[f(\tilde{B}_{H_i}(t_j),\ i=1,2,\ 1\le j\le n)\right] = \mathrm{E}\left[f(B_{H_i}(t_j),\ i=1,2,\ 1\le j\le n)\right].$$

Hence the finite dimensional Q–laws of $(\tilde{B}_{H_1}, \tilde{B}_{H_2})$ coincide with those of (B_{H_1}, B_{H_2}) under P. □

The same reasoning shows that under Q, the couple (X,Y) has the same law as $(\tilde{B}_{H_1}, \tilde{B}_{H_2})$. At this point, it is classical to apply the Itô formula (provided that it exists hence we need to assume here that $H_1 > 1/2$) to the product $f(X_t)\Lambda_t^h$ and then compute the conditional expectations to obtain the so–called unnormalized filter $\sigma_t(f) \overset{not}{=} \mathrm{E}_{\mathrm{Q}}\left[f(X_t)\Lambda_t^h \,|\, Y_s, s\le t\right]$. The moral of the part of this work is that all the technical difficulties can be overcome but the final result is not satisfactory. Actually, one finds (see [6]) that the equation satisfied by $\sigma_t(f)$ is of the form :

$$d\sigma_t(f) = \sigma_t(f''\psi(X_{t]}))\, dt + \sigma_t(f.h)\, dB_{H_2}(t),,$$

where $\psi(X_{t]})$ is a measurable function of the whole sample–path of X up to time t. This difficulty can be seen as a consequence of the long–range dependence of the fractional Brownian motion B_{H_1}. To overcome it, we filter the infinite dimensional process $\mathcal{X}$, where $\mathcal{X}$ is the $\mathbb{W} = \mathcal{C}_0([0,1], \mathbf{R})$–valued process

$$\mathcal{X}_t \stackrel{def}{=} \int_0^t K_{H_1}(.,s)\, dB^1(s),$$

which is related to B_{H_1} by the identity $\mathcal{X}_t(t) = B_{H_1}(t)$.

Using the Itô formula of Kuo [17] for Banach valued processes, for sufficiently regular $F : \mathbb{W} \to \mathbf{R}$, we find that the unnormalized filter $\tilde{\sigma}_t(F) \stackrel{def}{=} \mathrm{E}_{\mathrm{Q}}\left[F(\mathcal{X}_t)\Lambda_t^h \,|\, \mathcal{Y}_t\right]$ solves the differential equation :

$$(18) \quad \tilde{\sigma}_t(F) = F(x_0) + \int_0^t \tilde{\sigma}_s(F.h \circ p_s)\, dB(s)^2 + \frac{1}{2}\int_0^t \tilde{\sigma}_s\Big((K_{H_i}(.,s))^*\, D^2_\alpha F(.)(K_{H_i}(.,s))\Big)\, ds,$$

where $p_s(x) = x(s)$ for any $x \in \mathbb{W}$ and $D^2_\alpha F$ is the second order Fréchet dérivative of F in the direction $\mathcal{B}_{\alpha,2}$. As an example of the efficiency of this method, for $h = \mathrm{Id}$, choose $F(x) = \exp(i\beta x(1))$ where $x(1)$ is the value at time 1 of the element x of $\mathbb{W}$, and set $\mathfrak{X}(t,\beta) = \tilde{\sigma}_t(F)$, we obtain the following linear partial stochastic differential equation :

$$d\mathfrak{X}(t,\beta) = -\beta^2 K_H(1,t)^2 \mathfrak{X}(t,\beta)\, dt - i\frac{\partial \mathfrak{X}}{\partial \beta}(t,\beta)\, dB_{H_2}(t),$$
$$\mathfrak{X}(0,\beta) = x_0.$$

It is then tricky but possible to find an explicit Gaussian solution of this equation and thus to obtain the Kalman–Bucy filter of X with respect to Y (see [15] for a related result in which W^2 is a standard Brownian motion).

Appendix A. Deterministic fractional calculus

For $f \in L^1([0,1])$, the left and right fractional integrals of f are defined by :

$$(I^\alpha_{0^+}f)(x) \overset{def}{=} \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt\ ,\ x \geq 0,$$
$$(I^\alpha_{b^-}f)(x) \overset{def}{=} \frac{1}{\Gamma(\alpha)}\int_x^b f(t)(t-x)^{\alpha-1}dt\ ,\ x \leq b,$$

where $\alpha > 0$ and $I^0 = \mathrm{Id}$. For any $\alpha \geq 0$, any $f \in L^p([0,1])$ and $g \in L^q([0,1])$ where $p^{-1} + q^{-1} \leq \alpha$, we have :

$$\int_0^1 f(s)(I^\alpha_{0^+}g)(s)\ ds = \int_0^1 (I^\alpha_{1^-}f)(s)g(s)\ ds. \tag{19}$$

The Besov space $I^\alpha_{0^+}(L^p) \overset{not}{=} \mathcal{B}_{\alpha,p}$ is usually equipped with the norm :

$$\|f\|_{\mathcal{B}_{\alpha,p}} = \|I^{-\alpha}_{0^+}f\|_{L^p}.$$

We then have the following continuity results (see [13, 29]) :

Proposition A.1.

i. *If* $0 < \alpha < 1$, $1 < p < 1/\alpha$, *then* $I^\alpha_{0^+}$ *is a bounded operator from* $L^p([0,1])$ *into* $L^q([0,1])$ *with* $q = p(1-\alpha p)^{-1}$.

ii. *For any* $0 < \alpha < 1$ *and any* $p \geq 1$, $\mathcal{B}_{\alpha,p}$ *is continuously embedded in* $\mathrm{Hol}(\alpha - 1/p)$ *provided that* $\alpha - 1/p > 0$. $\mathrm{Hol}(\nu)$ *denotes the space of Hölder–continuous functions, null at time* 0, *equipped with the usual norm :*

$$\|f\|_{\mathrm{Hol}(\nu)} = \sup_{t\neq s}\frac{|f(t)-f(s)|}{|t-s|^\nu}.$$

We formally denote by $\mathrm{Hol}_-(\nu)$ *the intersection of the spaces* $\mathrm{Hol}(\eta)$ *for all* $\eta < \nu$ *and by* $\mathrm{Hol}_+(\nu)$ *the union of the spaces* $\mathrm{Hol}(\eta)$ *for* $\eta > \nu$.

iii. *For any* $0 < \alpha < \beta < 1$, $\mathrm{Hol}(\beta)$ *is compactly embedded in* $\mathcal{B}_{\alpha,\infty}$.

By $I^{-\alpha}_{0^+}$, respectively $I^{-\alpha}_{1^-}$, we mean the inverse map of $I^\alpha_{0^+}$, respectively $I^\alpha_{1^-}$.

Appendix B. Malliavin calculus

We give here a very sketchy introduction to the Malliavin calculus. We denote by $\mathcal{C}_n$ the simplex of $\mathbf{R}^n$, i.e.,

$$\mathcal{C}_n = \{(t_1,\ldots,t_n):\ 0 \leq t_1\ \leq t_2 \leq \ldots \leq t_n \leq 1\}.$$

For a deterministic function f such that :

$$\int\cdots\int_{\mathcal{C}_n} f(t_1,\ldots,t_n)^2\,dt_1\ldots dt_n\ <\ +\infty,$$

the n-th iterated Itô integral of f is given by :

$$I_n(f) \stackrel{def}{=} \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} f(t_1, \dots, t_n)^2 \, dB_{t_1} dB_{t_2} \dots dB_{t_n}.$$

We denote by $L^2_s([0,1]^n)$ the space of symmetric elements of $L^2([0,1])$. For $f \in L^2_s([0,1]^n)$, by definition we have :

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} f(t_1, \dots, t_n)^2 \, dB_{t_1} dB_{t_2} \dots dB_{t_n}.$$

It is well known that any square integrable random variable F on (Ω, P) has a chaos decomposition in $L^2(\Omega, \mathrm{P})$:

$$F = \sum_{n=0}^{+\infty} I_n(f_n), \tag{20}$$

where for each n, $f_n \in L^2_s([0,1]^n)$.

Definition B.1. *Let F be a square integrable random variable with a development of the form (20). The functional F is said to belong to $\mathbb{D}_{2,1}$ if and only if*

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2([0,1])} < +\infty, \tag{21}$$

and in this case we set :

$$\nabla_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(., t)),$$

and the sum of the series in (21) coincides with $E\left[\int_0^1 \nabla_t F^2 \, dt\right]$.

For $k > 1$, the k-th iterated Gross-Sobolev derivative of F, denoted $\nabla^{(k)} F$, is defined analogously by induction on k. We denote by $\mathbb{D}_{2,k}$ the set of functionals for which $\nabla^{(k)} F$ is well defined.

The following definition is specific to our context since we need to use a damped derivative which we will denote by D. Remind that $\mathcal{K} f = (K f)'$.

Definition B.2. *A process $u \in L^2(\Omega \times [0,1])$ will be said to be regular if the linear map $\mathcal{K}\nabla u$ is of trace class, i.e.,*

$$E\left[\left| \int_0^1 \mathcal{K} \circ \nabla_s u(s) \, ds \right| \right] < \infty.$$

∇ is a closed, unbounded and densely defined operator from $L^2(\Omega, \mathrm{P})$ in $L^2(\Omega \times [0,1])$. It thus admit an adjoint, denoted by δ, which is an unbounded operator on $L^2(\Omega \times [0,1])$ with values in $L^2(\Omega, \mathrm{P})$. δ is the so-called Skohorod integral or the divergence operator.

Definition B.3. *The domain of* δ *is the set of processes* $u \in L^2(\Omega \times [0,1])$ *such that :*

$$|E\left[\int_0^1 \nabla_t F u_t \, dt\right]| \le c\|F\|_{L^2(\Omega)},$$

for all $F \in \mathbb{D}_{2,1}$*, where* c *is some constant depending on* u*. If* u *belongs to* $\operatorname{Dom}\delta$*, then* $\delta(u)$ *is the element of* $L^2(\Omega, P)$ *characterized by :*

$$E[F\delta(u)] = E\left[\int_0^1 \nabla_t F u_t \, dt\right].$$

Moreover, it can be shown that the chaos expansion can be generalized to processes in the following sense (see [30]) :

Lemma B.1. *For* $u \in L^2(\Omega \times [0,1])$*, there exists a family of deterministic measurable and square integrable kernels* $f_m(t_1, \ldots, t_m, t)$ *is symmetric with respect to its first* m *variables and*

$$u_t = \sum_{n=0}^{\infty} I_n(f_n(.,t)), \tag{22}$$

where the convergence holds in $L^2(\Omega \times [0,1])$ *and*

$$E\left[\int_0^1 u_t^2 \, dt\right] = \sum_{n=0}^{\infty} n!\|f_n\|_{L^2([0,1]^{n+1})}.$$

We can now express the action of δ in terms of the Wiener chaos expansion.

Theorem B.1. *Let* $u \in L^2(\Omega \times [0,1])$ *with the expansion (22), then* u *belongs to* $\operatorname{Dom}\delta$ *if and only if*

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

converges in $L^2(\Omega, P)$*, where* $\tilde{f}_n$ *is the symmetrization of* f_n*, i.e.,*

$$\tilde{f}_n(t_1, \ldots, t_n, t) = \frac{1}{n+1}\Big[f_n(t_1, \ldots, t_n, t) + \sum_{i=1}^{n} f_n(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n)\Big].$$

Since the Skohorod integral in an extension of the Itô integral to a class of non-adapted processes, it is customary to choose a similar notation for the two integrals. That explains why we chose to denote in this text $\delta(u)$ by $\int u(s) \circ dB(s)$.

Definition B.4. *We define the "damped" gradient* D *by:*

$$DF(s) = \mathcal{K}(\nabla_. F)(s),$$

provided that the right-hand-side is meaningful – see the definition of the domain of $\mathcal{K}$ *in Remark 3.2.*

It is a common problem, in the standard case, to define properly trace$(\nabla u) = \int_0^1 \nabla_s u_s$ for a process u. Actually the gradient of a random variable is only a square integrable function and also is u_s; hence, in general, we don't have any information on the regularity of the function $s \mapsto \nabla_s u_s$. Not surprisingly, the sufficient conditions known for the trace to exist impose the continuity of this function (see [23]). Here the situation is on one side ($H > 1/2$) better and on the other side, worse. For $H > 1/2$, the regularizing effect of $\mathcal{K}$ entails that it is sufficient that u satisfies a (strong) integrability (see Theorem 3.6, all the more stringent as H is closer to $1/2$. On the other hand, for $H < 1/2$, the unboundedness of $\mathcal{K}$ entails a stronger condition on u than the continuity of ∇u, namely we expect $\nabla_. u$ to be $(1-H)$-Hölder (see Theorem 3.7).

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