Chapter 3
Wiener chaos

3.1 Chaos decomposition

Definition 3.1 (Iterated integrals on a simplex). For $t \in (0, 1]$, let

$$\mathfrak{C}_n(t) = \{(t_1, \cdots, t_n) \in [0,1]^n,\ 0 \le t_1 < \ldots < t_n \le t\}.$$

For $f \in L^2(\mathfrak{C}_n(t) \to \mathbf{R};\ \lambda)$, set

$$J_n(f)(t) = \int_0^t \, \mathrm{d}B(t_n) \int_0^{t_n} \, \mathrm{d}B(t_{n-1}) \ldots \int_0^{t_2} f(t_1, \cdots, t_n) \, \mathrm{d}B(t_1),$$

where the integrals are Itô integrals. For the sake of notations, set $\mathfrak{C}_n = \mathfrak{C}_n(1)$ and $J_n(f) = J_n(f)(1)$.

Remark 3.1. The structure of $\mathfrak{C}_n(t)$ ensures that at each internal integral, the integrand is adapted. Moreover,

$$J_n(f)(t) = \int_0^t J_{n-1}(f(., t_n)) \, \mathrm{d}B(t_n).$$

The Itô isometry then entails that

$$\mathbf{E}\left[J_n(f)J_m(g)\right] = \begin{cases} 0 & \text{if } n \ne m \\ \int_{\mathfrak{C}_n(t)} fg \, \mathrm{d}\lambda & \text{if } n = m. \end{cases} \tag{3.1}$$

We wish to extend this notion of iterated integral to function defined on the whole cube $[0,1]^n$ but we cannot get rid of the adaptability condition. It is then crucial to remark that for $f : [0,1]^n \to \mathbf{R}$ symmetric,

$$\int_{[0,1]^n} f \, \mathrm{d}\lambda = n! \int_{\mathfrak{C}_n} f \, \mathrm{d}\lambda,$$

since for any permutation σ of $\{1,\cdots,n\}$, the integral of f on $\mathfrak{C}_n$ is equal to its integral on

$$\sigma\mathfrak{C}_n = \left\{(t_1,\cdots,t_n)\in[0,1]^n,\ 0\le t_{\sigma(1)}<\ldots<t_{\sigma(n)}\le 1\right\}.$$

This motivates the following definition of the iterated integral:

Definition 3.2 (Generalized iterated integrals). Let $L_s^2 = L_s^2([0,1]^n \to \mathbf{R};\,\lambda)$ be the set of symmetric functions on $[0,1]^n$, square integrable with respect to the Lenbesgue measure. For $f \in L_s^2$,

$$J_n^s(f) = n!\, J_n(f\mathbf{1}_{\mathfrak{C}_n}).$$

If f belongs to $L^2([0,1]^n \to \mathbf{R};\,\lambda)$ but is not necessarily symmetric,

$$J_n^s(f) = J_n^s(f^s),$$

where f^s is the symmetrization of f:

$$f^s(t_1,\cdots,t_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} f(t_{\sigma(1)},\cdots,t_{\sigma(n)}).$$

In view of Eqn. (3.1), for $f,g \in L_s^2$, we have

$$\mathbf{E}\left[J_n^s(f)J_m^s(g)\right] = \begin{cases} 0 & \text{if } n\neq m \\ (n!)^2\displaystyle\int_{\mathfrak{C}_n} fg\ \mathrm{d}\lambda = n!\int_{[0,1]^n} fg\ \mathrm{d}\lambda & \text{if } n=m. \end{cases} \tag{3.2}$$

Theorem 3.1 (Chaos expansion of Doléans exponentials). *Let h belongs to $\mathcal{H}$. Then,*

$$\Lambda_h = 1 + \sum_{n=1}^{\infty} J_n(\dot{h}^{\otimes n}\mathbf{1}_{\mathfrak{C}_n}) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}J_n^s(\dot{h}^{\otimes n}),$$

where the convergence holds in $L^2(\mathcal{W}\to\mathbf{R};\,\mu)$.

Proof. Let

$$\Lambda_h(t) = \exp\left(\int_0^t \dot{h}(s)\ \mathrm{d}B(s) - \frac{1}{2}\int_0^1 \dot{h}(s)^2\ \mathrm{d}s\right).$$

The Itô calculus says that

$$\Lambda_h(t) = 1 + \int_0^t \Lambda_h(s)\,\dot{h}(s)\ \mathrm{d}B(s),$$

hence

$$\begin{aligned}
\Lambda_h(t) &= 1 + \int_0^t \Lambda_h(s)\,\dot{h}(s)\ \mathrm{d}B(s) \\
&= 1 + \int_0^t \left(1 + \int_0^s \Lambda_h(r)\dot{h}(r)\ \mathrm{d}B(r)\right))\,\dot{h}(s)\ \mathrm{d}B(s) \\
&= 1 + \int_0^t \dot{h}(s)\ \mathrm{d}B(s) + \int_0^t \left(\int_0^s \Lambda_h(r)\dot{h}(s)\dot{h}(r)\ \mathrm{d}B(r)\right)\ \mathrm{d}B(s) \\
&= 1 + \sum_{k=1}^n J_n(\dot{h}^{\otimes n}\mathbf{1}_{\mathfrak{C}_n}) + \int_{\mathfrak{C}_n}\prod_{j=1}^n \dot{h}(s_j)\ \Lambda_h(s_1)\ \mathrm{d}B(s_1)\ldots\ \mathrm{d}B(s_n) \\
&= 1 + \sum_{k=1}^n J_n(\dot{h}^{\otimes n}\mathbf{1}_{\mathfrak{C}_n}) + R_n.
\end{aligned}$$

It thus remains to show that R_n tends to 0 as n goes to infinity. According to (3.1),

$$\mathbf{E}\left[R_n^2\right] = \int_{\mathfrak{C}_n}\prod_{j=1}^n \dot{h}(s_j)^2\mathbf{E}\left[\Lambda_h(s_n)^2\right]\ \mathrm{d}s_1\ldots\ \mathrm{d}s_n. \tag{3.3}$$

Moreover,

$$\begin{aligned}
\mathbf{E}\left[\Lambda_h(s)^2\right] = \mathbf{E}&\left[\exp\left(2\int_0^s \dot{h}(u)\ \mathrm{d}B(u) - \int_0^s \dot{h}^2(u)\ \mathrm{d}u\right)\right] \\
&= \mathbf{E}\left[\Lambda_{2h}(s)\right]\exp(\|h\|_{\mathcal{H}}^2) = \exp(\|h\|_{\mathcal{H}}^2).
\end{aligned}$$

Plug this new expression into Eqn. (3.3) to obtain

$$\begin{aligned}
&\mathbf{E}\left[R_n^2\right] = \exp(\|h\|_{\mathcal{H}}^2)\ \int_{\mathfrak{C}_n}\prod_{j=1}^n \dot{h}(s_j)^2\ \mathrm{d}s_1\ldots\ \mathrm{d}s_n \\
&= \exp(\|h\|_{\mathcal{H}}^2)\ \frac{1}{n!}\int_{[0,1]^n} \dot{h}(s_j)^2\ \mathrm{d}s_1\ldots\ \mathrm{d}s_n = \exp(\|h\|_{\mathcal{H}}^2)\ \frac{1}{n!}\|h\|_{\mathcal{H}}^{2n} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

The result follows.

Definition 3.3 (Fock space). The Fock space $\mathfrak{F}_\mu(\mathcal{H})$ is the completion of the direct sum of the tensor powers of $\mathcal{H}$:

$$\mathfrak{F}_\mu(\mathcal{H}) = \mathbf{R} \oplus \bigoplus_{n=1}^\infty \mathcal{H}^{\otimes n}.$$

It is an Hilbert space when equipped with the norm

$$\|\oplus_{n=0}^\infty h_n\|_{\mathfrak{F}_\mu(\mathcal{H})}^2 = \sum_{n=0}^\infty \frac{1}{n!}\|h_n\|_{\mathcal{H}^{\otimes n}}^2.$$

Theorem 2.2 says that the set of tensor products is dense in $\mathcal{H}^{\otimes n}$, hence for a continuous linear map A from $\mathcal{H}$ into itself, we can define its tensor power on $\mathcal{H}^{\otimes n}$ by the rule

$$\begin{aligned} A^{\otimes n} : \mathcal{H}^{\otimes n} &\longrightarrow \mathcal{H}^{\otimes n} \\ \otimes_{j=1}^{n} h_j &\longmapsto \otimes_{j=1}^{n} A h_j . \end{aligned}$$

For an arbitrary element h of $\mathcal{H}^{\otimes n}$, the value of $A^{\otimes n} h$ is defined by a limiting procedure.

Definition 3.4 (Second quantization). The second quantization of A is the map from $\mathfrak{F}_\mu(\mathcal{H})$ into itself which coincides with $A^{\otimes n}$ on the n-th chaos.

Theorem 3.2. *The map*

$$\begin{aligned} \Upsilon : \mathcal{E} \subset L^2(\mathcal{W} \to \mathbf{R};\, \mu) &\longrightarrow \mathfrak{F}_\mu(\mathcal{H}) \\ F &\longmapsto \bigoplus_{n=0}^{\infty} \mathbf{E}\left[\nabla^{(n)} F\right], \end{aligned}$$

admits a continuous extension defined on $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$. We denote by $\Upsilon_n F$, the n-th term of the right-hand-side: $\Upsilon_n F = \mathbf{E}\left[\nabla^{(n)} F\right]$.

Proof. Remark that for $F = \Lambda_h \in \mathcal{E}$,

$$\nabla^{(n)} F = F\, h^{\otimes n}, \text{ hence } \mathbf{E}\left[\nabla^{(n)} F\right] = h^{\otimes n},$$

so we have

$$F = \mathbf{E}\left[F\right] + \sum_{n=1}^{\infty} \frac{1}{n!}\; J_n^s\big(\widetilde{\mathbf{E}\left[\dot{\nabla}^{(n)} F\right]}\big).$$

Since the chaos are orthogonal,

$$\begin{aligned} \mathbf{E}\left[F^2\right] \geq \mathbf{E}\left[(F - \mathbf{E}\left[F\right])^2\right] &= \sum_{n=1}^{\infty} \frac{1}{n!^2} \mathbf{E}\left[J_n^s\big(\widetilde{\mathbf{E}\left[\dot{\nabla}^{(n)} F\right]}\big)^2 \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left\| \widetilde{\mathbf{E}\left[\dot{\nabla}^{(n)} F\right]} \right\|^2_{L^2([0,1]^n \to \mathbf{R};\, \lambda^{\otimes n})} = \sum_{n=1}^{\infty} \frac{1}{n!} \left\| \mathbf{E}\left[\nabla^{(n)} F\right] \right\|^2_{\mathcal{H}^{\otimes n}} . \end{aligned}$$

Thus, by linearity, for any $F \in \mathcal{E}$,

$$\|\Upsilon F\|_{\mathfrak{F}_\mu(\mathcal{H})} \leq \|F\|_{L^2(\mathcal{W} \to \mathbf{R};\, \mu)}. \tag{3.4}$$

If $(F_n,\, n \geq 1)$ is a sequence of elements of $\mathcal{E}$ which converges to F in $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$, the sequence $(\Upsilon F_n,\, n \geq 1)$ is Cauchy in the Hilbert space $\mathfrak{F}_\mu(\mathcal{H})$, hence convergent. Then, ΥF can be unambiguously defined as $\lim_{n\to\infty} \Upsilon F_n$ and (3.4) holds for any $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$.

Theorem 3.3 (Chaos decomposition). *For any $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$,*

$$F = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{1}{n!}\, J_n^s\big(\widetilde{\dot{\Upsilon_n F}}\big). \tag{3.5}$$

This can be formally written as

$$F = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{1}{n!}\, J_n^s\Big(\mathbf{E}\,\widetilde{\dot{\big[\nabla^{(n)} F\big]}}\Big),$$

keeping in mind that $\mathbf{E}\left[\nabla^{(n)} F\right]$ is defined through Υ for general random variables.

The chaos decomposition means that $\mathfrak{F}_\mu(\mathcal{H})$ is isometrically isomorphic to $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$.

Proof. Without loss of generality, we may assume that $\mathbf{E}[F] = 0$. For $F = \Lambda_h - 1$, we know that

$$\nabla^{(n)} F = F\, h^{\otimes n}, \text{ hence } \Upsilon_n F = h^{\otimes n}.$$

Then, Theorem 3.1 means that Eqn. (3.5) holds true for $F = \Lambda_h - 1$ for any $h \in \mathcal{H}$. By linearity of the maps Υ_n, it is still true for linear combination of such random variables. Let $(F_k,\, k \geq 1)$ a sequence of elements of $\mathcal{E}$ converging to F in $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$. Since Υ is continuous in $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$,

$$\Upsilon F_k \xrightarrow[L^2(\mathcal{W}\to\mathbf{R};\,\mu)]{k\to\infty} \Upsilon F.$$

Since the chaos are orthogonal in $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$

$$\begin{aligned}\mathbf{E}\left[\left|\sum_{n=1}^{\infty} \frac{1}{n!}\, J_n^s\big(\Upsilon_n F_k\big) - \sum_{n=1}^{\infty} \frac{1}{n!}\, J_n^s\big(\Upsilon_n F\big)\right|^2\right] &= \sum_{n=1}^{\infty} \frac{1}{n!}\, \mathbf{E}\left[|\Upsilon_n F_k - \Upsilon_n F|^2\right] \\ &= \|\Upsilon(F_k - F)\|^2_{\oplus_{n=0}^{\infty}\mathcal{H}^{\otimes n}}.\end{aligned}$$

This means that

$$0 = F_k - \sum_{n=0}^{\infty} \frac{1}{n!}\, J_n^s\big(\Upsilon_n F_k\big) \xrightarrow{k\to\infty} F - \sum_{n=0}^{\infty} \frac{1}{n!}\, J_n^s\big(\Upsilon_n F\big).$$

The proof is thus complete.

Theorem 3.4 (Iterated integrals and iterated divergence). *For any $h \in \mathcal{H}$,*

$$J_n^s(\dot{h}^{\otimes n}) = \delta^n h^{\otimes n}.$$

Hence, for any $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$,

$$F = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{1}{n!}\, \delta^n\big(\Upsilon_n F\big). \tag{3.6}$$

Proof. For $F \in \mathcal{E}$, the Taylor-MacLaurin formula says that

$$F(\omega + \tau h) = F(\omega) + \sum_{n=1}^{\infty} \frac{\tau^n}{n!} \left\langle \nabla^{(n)} F(\omega),\, h^{\otimes n} \right\rangle_{\mathcal{H}^{\otimes n}}.$$

Hence,

$$\begin{aligned}\mathbf{E}[F(\omega+\tau h)] = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{\tau^n}{n!} \left\langle \Upsilon_n F,\, h^{\otimes n}\right\rangle_{\mathcal{H}^{\otimes n}} \\ = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{\tau^n}{n!} \mathbf{E}\left[F \delta^n h^{\otimes n}\right].\end{aligned}$$

On the other hand, the Cameron-Martin theorem and Theorem 3.1 induce that

$$\begin{aligned}\mathbf{E}[F(\omega+\tau h)] = \mathbf{E}[F\,\Lambda_{\tau h}] = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{E}\left[F\, J_n^s((\tau\dot{h})^{\otimes n})\right] \\ = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{\tau^n}{n!} \mathbf{E}\left[F\, J_n^s(\dot{h}^{\otimes n})\right].\end{aligned}$$

The result follows by identification of the coefficient of the two power series.

The very same method of identification can be used to prove the next results.

Theorem 3.5 (Gradient and conditional expectation). *For any* $t \in [0,1]$*, for any* $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$*,*

$$\mathbf{E}[F \,|\, \mathcal{F}_t] = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{1}{n!}\, \delta^n\Big(\Gamma_{\pi_t} \Upsilon_n F\Big) \tag{3.7}$$

where we recall that π_t *is the projection map*

$$\begin{aligned}\pi_t \,:\, \mathcal{H} &\longrightarrow \mathcal{H} \\ h &\longmapsto I^1(\dot{h}\,\mathbf{1}_{[0,t]}).\end{aligned}$$

Proof. The well known identity

$$\mathbf{E}\left[\exp\left(\int_0^1 \dot{h}(s)\ \mathrm{d}B(s) - \frac{1}{2}\int_0^1 \dot{h}(s)^2\ \mathrm{d}s\right) \mid \mathcal{F}_t\right] = \exp\left(\int_0^t \dot{h}(s)\ \mathrm{d}B(s) - \frac{1}{2}\int_0^t \dot{h}(s)^2\ \mathrm{d}s\right)$$

can be written as

$$\mathbf{E}\left[\Lambda_h \mid \mathcal{F}_t\right] = \Lambda_{\pi_t h}.$$

Apply this equality to τh and consider the chaos expansion of both terms. Since the convergence of the series holds in $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$, we can apply Fubini's theorem straightforwardly.

$$1 + \sum_{n=1}^{\infty} \frac{\tau^n}{n!}\, \mathbf{E}\left[\delta^n h^{\otimes n} \mid \mathcal{F}_t\right] = 1 + \sum_{n=1}^{\infty} \frac{\tau^n}{n!} \delta^n(\pi_t^{\otimes n} h^{\otimes n}).$$

This means that (3.7) holds for $F \in \mathcal{E}$ and by density, it is true for any $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$.

Lemma 3.1. *The map*

$$\begin{aligned} \partial_{\mathcal{W}}\ :\ \mathcal{E} \subset L^2(\mathcal{W} \to \mathbf{R};\, \mu) &\longrightarrow L^2(\mathcal{W} \times [0,1] \to \mathbf{R};\, \mu \otimes \lambda) \\ F &\longmapsto \left(s \mapsto \mathbf{E}\left[\dot{\nabla}_s F \mid \mathcal{F}_s\right], s \in [0,1]\right) \end{aligned}$$

can be extended as a continuous map from $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$ into $L^2(\mathcal{W} \times [0,1] \to \mathbf{R};\, \mu \otimes \lambda)$.

Proof. For $F = \Lambda_k \in \mathcal{E}$,

$$\partial_{\mathcal{W}} F = \left(s \longmapsto \Lambda_k(s) \dot{k}(s)\right)$$

and

$$F = 1 + \int_0^1 \partial_{\mathcal{W}} F(s)\ \mathrm{d}B(s).$$

On the one hand, Itô isometry yields

$$\mathbf{E}\left[\left(\int_0^1 \partial_{\mathcal{W}} F(s)\ \mathrm{d}B(s)\right)^2\right] = \|\partial_{\mathcal{W}} F\|^2_{L^2(\mathcal{W} \times [0,1])}. \tag{3.8}$$

On the other hand,

$$\mathbf{E}\left[\left(\int_0^1 \partial_{\mathcal{W}} F(s)\ \mathrm{d}B(s)\right)^2\right] = \mathbf{E}\left[(F-1)^2\right] = \mathbf{E}\left[F^2\right] - 1 \le \mathbf{E}\left[F^2\right]. \tag{3.9}$$

Let $F \in L^2(\mathcal{W})$ be the limit of $(F_n, n \geq 1)$ a sequence of elements of $\mathcal{E}$. Eqn. (3.8) and (3.9) imply that $(\partial_{\mathcal{W}} F_n, n \geq 1)$ is Cauchy in $L^2(\mathcal{W}\times[0,1], \mu\otimes\lambda)$, hence convergent to a limit, we define to be $\partial_{\mathcal{W}} F$.

Theorem 3.6 (Itô-Clark-Ocone-Üstünel formula). *For $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$,*

$$F = \mathbf{E}\left[F\right] + \delta\big(\, I^1(\partial_{\mathcal{W}} F)\,\big). \tag{3.10}$$

For $F \in \mathbb{D}_{1,2}$, this boils down to

$$F = \mathbf{E}\left[F\right] + \int_0^1 \mathbf{E}\left[\dot{\nabla}_s F \,|\, \mathcal{F}_s\right]\ dB(s). \tag{3.11}$$

Proof. For $F = \Lambda_k \in \mathcal{E}$, recall that $\dot{\nabla} F = \dot{k}$, thus we do have

$$F = \mathbf{E}\left[F\right] + \int_0^1 \Lambda_k(s)\dot{k}(s)\ \mathrm{d}B(s) = \mathbf{E}\left[F\right] + \int_0^1 \mathbf{E}\left[\dot{\nabla}_s F \,|\, \mathcal{F}_s\right]\ \mathrm{d}B(s).$$

The proof follows by the density of $\mathcal{E}$ in $L^2(\mathcal{W} \to \mathbf{R};\, \mu)$ and Lemma 3.1.

Lemma 3.2. *Then, the set of pure tensors $\dot{h}^{\otimes n}$ for $h \in L^2([0,1] \to \mathbf{R};\, \lambda)$ is dense in $L^2_s([0,1]^n \to \mathbf{R};\, \lambda^{\otimes n})$.*

Proof. We already know (see Theorem 2.2) that tensor products $\dot{h}_1 \otimes \ldots \otimes \dot{h}_n$ with $h_i \in L^2([0,1] \to \mathbf{R};\, \lambda)$ are dense in $L^2([0,1]^n \to \mathbf{R};\, \lambda^{\otimes n})$ and that the symmetrization operation is continuous from $L^2([0,1]^n \to \mathbf{R};\, \lambda)$ into $L^2_s([0,1]^n \to \mathbf{R};\, \lambda^{\otimes n})$. Apply the symmetrization to any approximating sequence to obtain a sequence of linear combinations of pure tensors which converges to the symmetrization of f_n, which is already f_n.

Theorem 3.7 (Gradient of chaos). *For $\dot{h}_n \in L^2([0,1]^n \to \mathbf{R};\, \mu)$, let $\dot{h}(.,r)$ be the element of L^2_s defined by*

$$\begin{aligned} \dot{h}_n(.,r)\,:\, [0,1]^{n-1} &\longrightarrow \mathbf{R} \\ (s_1, \cdots, s_{n-1}) &\longmapsto \dot{h}_n(s_1, \cdots, s_{n-1}, r). \end{aligned}$$

Then,

$$\dot{\nabla}_r J^s_n(\dot{h}_n)(t) = n\, J^s_{n-1}(\dot{h}_n(.,r)). \tag{3.12}$$

Proof. In view of Lemma 3.2, it is sufficient to prove (3.12) for $\dot{h}_n = \dot{h}^{\otimes n}$. It boils down to prove

$$\dot{\nabla}_r J^s_n(\dot{h}^{\otimes n}) = n\, J^s_{n-1}(\dot{h}^{\otimes n-1})\dot{h}.$$

Let $h \in \mathcal{H}$, we already know that Λ_h belongs to $\mathbb{D}_{1,2}$ and that $\nabla \Lambda_h = \Lambda_h\, h$. Apply this reasoning to τh:

$$\nabla\left(\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\ J_n^s(\dot{h}^{\otimes n})\right)=\sum_{n=0}^{\infty}\frac{\tau^{n+1}}{n!}\ J_n^s(\dot{h}^{\otimes n})\,h=\sum_{n=1}^{\infty}\frac{\tau^n}{(n-1)!}\ J_{n-1}^s(\dot{h}^{\otimes n-1})\,h$$
$$=\sum_{n=1}^{\infty}\frac{\tau^n}{n!}\ n\,J_{n-1}^s(\dot{h}^{\otimes n-1})\,h.$$

Furthermore, if

$$\Lambda_h^{(N)}=\sum_{n=0}^{N}\frac{1}{n!}\ J_n^s(\dot{h}^{\otimes n}),$$

it holds that

$$\Lambda_h^{(N)}\xrightarrow[L^2(\mathcal{W})]{N\to\infty}\Lambda_h.$$

Consequently, $(\nabla\Lambda_h^{(N)},\ n\geq 1)$ converges weakly in $\mathbb{D}_{1,2}(\mathcal{H})$ to $\nabla\Lambda_h$: For $U\in\mathbb{D}_{1,2}(\mathcal{H})\subset\operatorname{Dom}_2\delta$,

$$\mathbf{E}\left[\left\langle\nabla\Lambda_h^{(N)},\,U\right\rangle_{\mathcal{H}}\right]=\mathbf{E}\left[\Lambda_h^{(N)}\,\delta U\right]\xrightarrow{N\to\infty}\mathbf{E}\left[\Lambda_h\,\delta U\right]=\mathbf{E}\left[\langle\nabla\Lambda_h,\,U\rangle_{\mathcal{H}}\right].$$

Furthermore,

$$\mathbf{E}\left[\left\langle\nabla\Lambda_{\tau h}^{(N)},\,U\right\rangle_{\mathcal{H}}\right]=\sum_{n=1}^{N}\frac{\tau^n}{n!}\mathbf{E}\left[\left\langle\nabla J_n^s(\dot{h}^{\otimes n}),\,U\right\rangle_{\mathcal{H}}\right],$$

hence,

$$\sum_{n=1}^{\infty}\frac{\tau^n}{n!}\mathbf{E}\left[\left\langle\nabla J_n^s(\dot{h}^{\otimes n}),\,U\right\rangle_{\mathcal{H}}\right]=\sum_{n=1}^{\infty}\frac{\tau^n}{n!}n\,\mathbf{E}\left[\left\langle J_{n-1}^s(\dot{h}^{\otimes n-1})\,h,\,U\right\rangle_{\mathcal{H}}\right].$$

Identify the coeeficient of τ^n: For any $U\in\mathbb{D}_{1,2}(\mathcal{H})$

$$\mathbf{E}\left[\left\langle\nabla J_n^s(\dot{h}^{\otimes n}),\,U\right\rangle_{\mathcal{H}}\right]=n\,\mathbf{E}\left[\left\langle J_{n-1}^s(\dot{h}^{\otimes n-1})\,h,\,U\right\rangle_{\mathcal{H}}\right],$$

hence the result.

Corollary 3.1. *A random variable $F\in L^2(\mathcal{W})$ belongs to $\mathbb{D}_{2,1}$ if and only if*

$$\sum_{n=1}^{\infty}n^2\,\|\Upsilon_nF\|^2_{L^2_s([0,1]^n)}<\infty.$$

Definition 3.5. For $\dot{f}\in L^2([0,1]^n\to\mathbf{R};\,\lambda^{\otimes n})$ and $\dot{g}\in L^2([0,1]^m\to\mathbf{R};\,\lambda^{\otimes n})$, for $i\leq n\wedge m$, the i-th contraction of $\dot{f}$ and $\dot{g}$ is defined by

$$(\dot f \otimes_i \dot g)(t_1, \cdots, t_{n-i}, s_1, \cdots, s_{m-i})$$
$$= \int_{[0,1]^i} \dot f(t_1, \cdots, t_{n-i}, u_1, \cdots, u_i)\, \dot g(s_1, \cdots, s_{m-i}, u_1, \cdots, u_i)\; \mathrm{d}u_1 \ldots u_i.$$

It is an element of $L^2([0,1]^{n+m-2i})$. Its symmetrization is denoted by $\dot f \overset{s}{\otimes}_i g$.

Theorem 3.8 (Multiplication of iterated integrals). *For* $\dot f \in L^2([0,1]^n \to \mathbf{R};\, \lambda^{\otimes n})$ *and* $\dot g \in L^2([0,1]^m \to \mathbf{R};\, \lambda^{\otimes n})$*,*

$$J_n^s(\dot f) J_m^s(\dot g) = \sum_{i=0}^{n\wedge m} \frac{n!m!}{i!(n-i)!(m-i)!}\; J_{n+m-2i}(\dot f \overset{s}{\otimes}_i \dot g). \tag{3.13}$$

Proof. We give the proof for $n=1$, the general case follows the same principle with much involved notations and computations (see [Üst14]).

For $\psi \in \mathcal{E}$,

$$\mathbf{E}\left[J_m^s(\dot g) J_1^s(\dot f)\psi\right] = \mathbf{E}\left[\delta^m(g)\,\delta(f)\psi\right] = \mathbf{E}\left[\left\langle \nabla^{(m)}(\psi\,\delta(f)),\, g\right\rangle_{\mathcal{H}^{\otimes m}}\right].$$

Recall that $\nabla\delta(f) = f$ and that $\nabla^k\delta(f) = 0$ if $k \ge 2$. The Leibniz formula then implies that

$$\nabla^{(m)}\big(\psi\,\delta(f)\big) = \delta(f)\,\nabla^{(m)}\psi + m\,\nabla^{(m-1)}\psi \otimes f.$$

On the one hand,

$$\begin{aligned}\mathbf{E}\left[\delta(f)\left\langle \nabla^{(m)}\psi,\, g\right\rangle_{\mathcal{H}^{\otimes m}}\right] &= \mathbf{E}\left[\delta f\;\left\langle \nabla^{(m)}\psi,\, g\right\rangle_{\mathcal{H}^{\otimes m}}\right]\\ &= \mathbf{E}\left[\left\langle \nabla^{(m+1)}\psi,\, g\otimes f\right\rangle_{\mathcal{H}^{\otimes(m+1)}}\right]\\ &= \mathbf{E}\left[\psi\;\delta^{m+1}(g \overset{s}{\otimes}_0 f)\right].\end{aligned}$$

On the other hand, a simple application of Fubini's Theorem yields

$$\mathbf{E}\left[\left\langle \dot\nabla^{(m-1)}\psi\otimes f,\, g\right\rangle_{\mathcal{H}^{\otimes m}}\right]$$
$$= \int_{[0,1]^m} \dot\nabla^{(m-1)}_{s_1,\cdots,s_{m-1}}\psi\, \dot f(s_m)\; \dot g(s_1,\cdots,s_m)\; \mathrm{d}s_1 \ldots\; \mathrm{d}s_m.$$

Since $\dot\nabla^{(m-1)}\psi$ and $\dot g$ are symmetric, we have

$$\int_{[0,1]^m} \dot\nabla^{(m-1)}_{s_1,\cdots,s_{m-1}}\psi\, \dot f(s_m)\; \dot g(s_1,\cdots,s_m)\; \mathrm{d}s_1 \ldots\; \mathrm{d}s_m$$
$$= \frac{1}{m!}\sum_{\tau\in\mathfrak{S}_m} \int_{[0,1]^m} \dot\nabla^{(m-1)}_{s_{\tau(1)},\cdots,s_{\tau(m-1)}}\psi\, \dot f(s_{\tau(m)})\; \dot g(s_{\tau(1)},\cdots,s_{\tau(m)})\; \mathrm{d}\mathfrak{s}$$

where $\mathrm{d}\mathfrak{s} = \mathrm{d}s_1 \dots \mathrm{d}s_m$. We partition $\mathfrak{S}_m$ into the m disjoints sets

$$\mathfrak{S}_m^j = \{\tau \in \mathfrak{S}_m, \tau(m) = j\}$$

We get

$$\begin{aligned}
&\sum_{\tau\in\mathfrak{S}_m} \int_{[0,1]^m} \dot{\nabla}^{(m-1)}_{s_{\tau(1)},\cdots,s_{\tau(m-1)}}\psi\, \dot{f}(s_{\tau(m)})\ \dot{g}(s_{\tau(1)},\cdots,s_{\tau(m)})\ \mathrm{d}\mathfrak{s} \\
&\quad = \sum_{j=1}^{m} \sum_{\tau\in\mathfrak{S}_m^j} \int_{[0,1]^{m-1}} \dot{\nabla}^{(m-1)}_{s_{\tau(1)},\cdots,s_{\tau(m-1)}}\psi \int_{[0,1]} \dot{f}(s)\ \dot{g}(s_{\tau(1)},\cdots,s)\ \mathrm{d}s\ \mathrm{d}\hat{\mathfrak{s}}_j,
\end{aligned} \tag{3.14}$$

where $\mathrm{d}\hat{\mathfrak{s}}_j = \mathrm{d}s_1 \dots \mathrm{d}s_{j-1}\ \mathrm{d}s_{j+1} \dots$ We can rewrite the last inner integral as

$$\int_{[0,1]^{m-1}} \dot{\nabla}^{(m-1)}_{s_{\tau(1)},\cdots,s_{\tau(m-1)}}\psi\ (\dot{f}\otimes_1 \dot{g})(s_{\tau(1)},\cdots,s_{\tau(m-1)})\ \mathrm{d}s_{\tau(1)}\dots\ \mathrm{d}s_{\tau(m-1)},$$

which makes apparent that this integral does not depend on j, hence it appears m times in (3.14). Since $\mathfrak{S}_m^j$ is in bijection with $\mathfrak{S}_{m-1}$, we obtain

$$\begin{aligned}
&\frac{1}{m!}\sum_{\tau\in\mathfrak{S}_m} \int_{[0,1]^m} \dot{\nabla}^{(m-1)}_{s_{\tau(1)},\cdots,s_{\tau(m-1)}}\psi\, \dot{f}(s_{\tau(m)})\ \dot{g}(s_{\tau(1)},\cdots,s_{\tau(m)})\ \mathrm{d}\mathfrak{s} \\
&= \frac{1}{(m-1)!}\sum_{\sigma\in\mathfrak{S}_{m-1}} \int_{[0,1]^{m-1}} \dot{\nabla}^{(m-1)}_{s_{\sigma(1)},\cdots,s_{\sigma(m-1)}}\psi\, (\dot{f}\otimes_1 \dot{g})(s_{\sigma(1)},\cdots,s_{\sigma(m-1)})\ \mathrm{d}\hat{\mathfrak{s}}_j \\
&= \int_{[0,1]^{m-1}} (\theta \overset{s}{\otimes}_1 \dot{f})(s_1,\cdots,s_{m-1})\ \mathrm{d}s_1\dots\ \mathrm{d}s_{m-1} \\
&= \int_{[0,1]^{m-1}} \dot{\nabla}^{(m-1)}_{s_1,\cdots,s_{m-1}}\psi\ (\dot{f} \overset{s}{\otimes}_1 \dot{g})(s_1,\cdots,s_{m-1})\ \mathrm{d}s_1\dots\ \mathrm{d}s_{m-1}
\end{aligned}$$

where

$$\theta\ :\ (s_1,\cdots,s_{m-1},s) \longmapsto \dot{\nabla}^{(m-1)}_{s_1,\cdots,s_{m-1}}\psi\ g(s_1,\cdots,s_{m-1},s).$$

Finally, we get

$$\begin{aligned}
&\mathbf{E}\left[\left\langle \dot{\nabla}^{(m-1)}\psi\otimes f,\, g\right\rangle_{\mathcal{H}^{\otimes m}}\right] \\
&\qquad = \mathbf{E}\left[\int_{[0,1]^m} \dot{\nabla}^{(m-1)}_{s_1,\cdots,s_{m-1}}\psi\, (\dot{f} \overset{s}{\otimes}_1 \dot{g})(s_1,\cdots,s_{m-1})\ \mathrm{d}s_1\dots\ \mathrm{d}s_m\right] \\
&\qquad\qquad = \mathbf{E}\left[\phi\, \delta^{m-1}(\dot{f} \overset{s}{\otimes}_1 \dot{g})\right].
\end{aligned}$$

The result follows by the density of $\mathcal{E}$ in $L^2(\mathcal{W})$.

Corollary 3.2 (Divergence on chaos). *Let*

$$\dot{U}(t) = \sum_{n=0}^{\infty} J_n^s(\dot{h}_n(.,t))$$

where $\dot{h}_n$ belongs to $L^2([0,1]^{n+1})$ and is symmetric with respect to its n first variables. Then,

$$\delta U = \sum_{n=0}^{\infty} J_{n+1}^s(\tilde{h}_n)$$

where

$$\tilde{h}_n(t_1, \cdots, t_n, t_{n+1}) = \frac{1}{n+1}\Big[\dot{h}_n(t_1, \cdots, t_n, t_{n+1}) \\ + \sum_{i=1}^{n} \dot{h}_n(t_1, \cdots, t_{i-1}, t_{n+1}, t_{i+1}, \cdots, t_i)\Big]. \quad (3.15)$$

Proof. As before, we reduce the problem to $\dot{h}_n(.,t) = \dot{h}^{\otimes n}\dot{g}(t)$. Then,

$$J_n^s(\dot{h}^{\otimes n}\dot{g}(t)) = J_n^s(\dot{h}^{\otimes n})\,\dot{g}(t).$$

Eqn. (2.11), (3.13) and (3.12) imply

$$\delta(J_n^s(\dot{h}^{\otimes n})\,\dot{g}) = J_n^s(\dot{h}^{\otimes n})J_1(\dot{g}) - \left\langle \nabla J_n^s(\dot{h}^{\otimes n}),\, g \right\rangle_{\mathcal{H}} \\ = J_{n+1}^s(\dot{h}^{\otimes n} \overset{s}{\otimes} \dot{g}) + nJ_{n-1}^s(\dot{h}^{\otimes n} \overset{s}{\otimes}_1 \dot{g}) - nJ_{n-1}^s(\dot{h}^{\otimes n-1}) \int_0^1 \dot{h}(s)\dot{g}(s)\, \mathrm{d}s. \quad (3.16)$$

By its very definition,

$$\big(\dot{h}^{\otimes n} \overset{s}{\otimes}_1 \dot{g}\big)(t_1, \cdots, t_{n-1}) = \prod_{j=1}^{n-1} \dot{h}(t_j) \int_0^1 \dot{h}(s)\dot{g}(s)\, \mathrm{d}s,$$

hence the last two terms of (3.16) do cancel each other. Since $\dot{h}^{\otimes n-1}$ is already symmetric, the symmetrization of $\dot{h}^{\otimes n-1} \otimes \dot{g}$ reduces to

$$\Big(\dot{h}^{\otimes n-1} \overset{s}{\otimes} \dot{g}\Big)(t_1, \cdots, t_{n+1}) \\ = \frac{1}{n+1}\left[\prod_{j=1}^{n} \dot{h}(t_j)\dot{g}(t_{n+1}) + \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \dot{h}(t_j)\dot{g}(t_i)\dot{h}(t_{n+1})\right],$$

which corresponds to (3.16) for general $\dot{h}_n$.

3.2 Ornstein-Uhlenbeck operator

In $\mathbf{R}^n$, the adjoint of the usual gradient is the divergence operator and the composition of divergence and gradient is the ordinary Laplacian. Since we have at our disposal, a notion of gradient and the corresponding divergence, we can consider the associated Laplacian, sometimes called Gross Laplacian, defined as

$$L = \delta\nabla.$$

A simple calculation shows the following which justifies the physicists' denomination of L as the number operator.

Theorem 3.9 (Number operator). *For $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$ of chaos decomposition*

$$F = \mathbf{E}\,[F] + \sum_{n=1}^{\infty} J_n^s(\dot{h}_n).$$

Then,

$$LF = \sum_{n=1}^{\infty} n\, J_n^s(\dot{h}_n).$$

The map L is invertible from $L_0^2 = \{F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu),\, \mathbf{E}\,[F] = 0\}$ into itself:

$$L^{-1}F = \sum_{n=1}^{\infty} \frac{1}{n}\, J_n^s(\dot{h}_n).$$

From there, it is customary to define the so-called Ornstein-Uhlenbeck operator from its action on chaos.

Definition 3.6 (Ornstein-Ulenbeck operator). Let $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$ of chaos decomposition

$$F = \mathbf{E}\,[F] + \sum_{n=1}^{\infty} J_n^s(\dot{h}_n).$$

For any $t > 0$,

$$P_t F = \mathbf{E}\,[F] + \sum_{n=1}^{\infty} e^{-nt}\, J_n^s(\dot{h}_n).$$

Formally, we can write $P_t = e^{-tL}$.

From these definitions, the following properties are straightforward

Theorem 3.10. *For any $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$, for any $s, t \geq 0$,*

$$P_{t+s}F = P_s(P_tF).$$

For any $F \in \mathbb{D}_{p,1}$,

$$\nabla P_tF = e^{-t}P_t\nabla F. \tag{3.17}$$

The Ornstein-Uhlenbeck can be alternatively defined by the so-called Mehler formula:

Theorem 3.11. *For any* $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$

$$P_tF(\omega) = \int_{\mathcal{W}} F(e^{-t}\omega + \sqrt{1-e^{-2t}}y)\ d\mu(y). \tag{3.18}$$

Its definition relies on the invariance by rotations of the Gaussian measure. In what follows, let $\beta_t = \sqrt{1-e^{-2t}}$.

Lemma 3.3. *For any* $t > 0$, *consider the transformation*

$$\begin{aligned} R_t\,:\, \mathcal{W}\times\mathcal{W} &\longrightarrow \mathcal{W}\times\mathcal{W} \\ (\omega,\eta) &\longmapsto \left(e^{-t}\omega + \beta_t\eta,\ -\beta_t\omega + e^{-t}\eta\right). \end{aligned}$$

Then the image of $\mu\otimes\mu$ *by* R_t *is still* $\mu\otimes\mu$.

Proof. Let h_1 and h_2 belong to $\mathcal{W}^*$. Then,

$$\begin{aligned} &\int_{\mathcal{W}\times\mathcal{W}} \exp\left(i\,\langle (h_1,h_2), R_t(\omega,\eta)\rangle_{\mathcal{W}^*\times\mathcal{W}^*,\mathcal{W}\times\mathcal{W}}\right)\ \mathrm{d}\mu(\omega)\ \mathrm{d}\mu(\eta) \\ &= \int_{\mathcal{W}} \exp\left(i\left\langle e^{-t}h_1 - \beta_th_2, \omega\right\rangle_{\mathcal{W}^*,\mathcal{W}}\right)\ \mathrm{d}\mu(\omega) \\ &\qquad\qquad \times \int_{\mathcal{W}} \exp\left(i\left\langle e^{-t}h_2 + \beta_th_1, \eta\right\rangle_{\mathcal{W}^*,\mathcal{W}}\right)\ \mathrm{d}\mu(\eta) \\ &= \exp\left(-\frac{1}{2}\left(\|e^{-t}h_1 - \beta_th_2\|_{\mathcal{H}}^2 + \|e^{-t}h_2 + \beta_th_1\|_{\mathcal{H}}^2\right)\right) \\ &\qquad\qquad = \exp\left(-\frac{1}{2}\|h_1\|_{\mathcal{H}}^2\right)\exp\left(-\frac{1}{2}\|h_2\|_{\mathcal{H}}^2\right). \end{aligned}$$

In view of the characterization of the Wiener measure, this completes the proof.

Proof (Proof of Theorem 3.11). We know that for $F \in L^2(W \to \mathbf{R};\, \mu)$,

$$\sum_{n=1}^{\infty} \frac{1}{n!}\mathbf{E}\left[J_n^s(\dot{h}_n)^2\right] < \infty.$$

If each kernel is multiplied by a constant smaller than 1, the convergence also holds, hence for any $t \geq 0$,

$$\|P_tF\|_{L^2(W\to\mathbf{R};\,\mu)} \le \|F\|_{L^2(W\to\mathbf{R};\,\mu)}.$$

Denote temporarily

$$T_tF(\omega) = \int_{\mathcal{W}} F(e^{-t}\omega + \sqrt{1-e^{-2t}}y)\ \mathrm{d}\mu(y).$$

We have

$$\begin{aligned}\int_{\mathcal{W}} T_tF(\omega)^2\ \mathrm{d}\mu(\omega) &= \int_{\mathcal{W}^2} F(e^{-t}\omega + \sqrt{1-e^{-2t}}y)^2\ \mathrm{d}\mu(\omega)\ \mathrm{d}\mu(y)\\ &= \int_{\mathcal{W}^2} \bar{F}\big(R_t(\omega,\eta)\big)^2\ \mathrm{d}\mu(\omega)\ \mathrm{d}\mu(y),\end{aligned}$$

where

$$\begin{aligned}\bar{F}\,:\,\mathcal{W}\times\mathcal{W} &\longrightarrow \mathbf{R}\\ (\omega,\eta) &\longmapsto F(\omega).\end{aligned}$$

According to Lemma 3.3,

$$\begin{aligned}\int_{\mathcal{W}^2} \bar{F}\big(R_t(\omega,\eta)\big)^2\ \mathrm{d}\mu(\omega)\ \mathrm{d}\mu(y) &= \int_{\mathcal{W}^2} \bar{F}(\omega,\eta)^2\ \mathrm{d}\mu(\omega)\ \mathrm{d}\mu(y)\\ &= \|F\|^2_{L^2(W\to\mathbf{R};\,\mu)}.\end{aligned}$$

This means it is sufficient to prove (3.18) for the elements of $\mathcal{E}$. By definition,

$$\delta h\big(e^{-t}\omega + \beta_t y\big) = \delta(e^{-t}h)(\omega) + \delta(\beta_t h)(y)$$

and

$$\|h\|^2_{\mathcal{H}} = \|e^{-t}h\|^2_{\mathcal{H}} + \|\beta_t h\|^2_{\mathcal{H}}.$$

Hence,

$$\Lambda_h\big(e^{-t}\omega + \beta_t y\big) = \Lambda_{e^{-t}h}(\omega)\Lambda_{\beta_t h}(y).$$

Hence,

$$\int_{\mathcal{W}} \Lambda_h\big(e^{-t}\omega + \beta_t y\big)\ \mathrm{d}\mu(y) = \Lambda_{e^{-t}h}(\omega)\int_{\mathcal{W}} \Lambda_{\beta_t h}(y)\ \mathrm{d}\mu(y) = \Lambda_{e^{-t}h}(\omega).$$

Now then, the chaos decomposition of $\Lambda_{e^{-t}h}(\omega)$ is given by

$$\Lambda_{e^{-t}h}(\omega) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} J_n^s\big((e^{-t}\dot{h})^{\otimes n}\big) = 1 + \sum_{n=1}^{\infty} \frac{e^{-nt}}{n!} J_n^s\big(\dot{h}^{\otimes n}\big).$$

The proof is thus complete.

Theorem 3.12. *The semi-group is ergodic and admits μ as stationary measure. As a consequence,*

$$\int_{\mathcal{W}} F \ d\mu - F = -\int_0^\infty LP_tF \ d\mu \tag{3.19}$$

and for F centered,

$$L^{-1}F = \int_0^\infty P_tF \ dt. \tag{3.20}$$

Proof. From the Mehler formula, we see by dominated convergence that

$$P_tF(\omega) \xrightarrow[\text{w.p.1}]{t\to\infty} \int_{\mathcal{W}} F \ \mathrm{d}\mu.$$

In view of Lemma 3.3,

$$\begin{aligned}\int_{\mathcal{W}} P_tF(\omega) \ \mathrm{d}\mu(\omega) &= \int_{\mathcal{W}^2} \bar{F}\big(R_t(\omega,\, y)\big) \ \mathrm{d}\mu(\omega) \ \mathrm{d}\mu(y) \\ &= \int_{\mathcal{W}^2} \bar{F}(\omega,\, y) \ \mathrm{d}\mu(\omega) \ \mathrm{d}\mu(y) = \int_{\mathcal{W}} F(\omega) \ \mathrm{d}\mu(\omega).\end{aligned}$$

This proves the stationarity of μ. Now, it comes from the chaos decomposition that

$$\frac{d}{dt}P_tF = -LP_tF,$$

hence

$$P_tF(\omega) - P_0F(\omega) = -\int_0^t LP_tF(\omega) \ \mathrm{d}t.$$

Let t go to infinity to obtain (3.19). Eqn. (3.20) is a direct consequence of the chaos decomposition.

The Mehler formula shows that P_tF is a convolution operator and as such has some strong regularization properties.

Theorem 3.13 (Regularization). *For $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$, for any $t > 0$, P_tF belongs to $\cap_{k\geq 1}\mathbb{D}_{k,p}$. Moreover,*

$$\left\langle \nabla^{(k)} P_tF,\, h \right\rangle_{\mathcal{H}^{\otimes k}} = \left(\frac{e^{-t}}{\beta_t}\right)^k \int_{\mathcal{W}} F(e^{-t}\omega + \beta_t y) \ \delta^k h(y) \ d\mu(y).$$

Proof. We give the proof for $k = 1$. The general situation is obtained by induction. For $F \in \mathcal{S}$,

$$\left\langle \nabla^{(k)} P_tF,\, h \right\rangle_{\mathcal{H}} = \frac{d}{d\varepsilon} P_tF(\omega + \varepsilon h)\bigg|_{\varepsilon=0}$$

The trick is then to consider that the translation by h operates not on ω but on y:

$$\begin{aligned} P_tF(\omega+\varepsilon h) &= \int_{\mathcal{W}} F\big(e^{-t}(\omega+\varepsilon h)+\beta_t y\big)\ \mathrm{d}\mu(y) \\ &= \int_{\mathcal{W}} F\big(e^{-t}\omega+\beta_t(y+\frac{\varepsilon e^{-t}}{\beta_t}h)\big)\ \mathrm{d}\mu(y). \end{aligned}$$

According to the Cameron-Martin (Theorem 1.8),

$$P_tF(\omega+\varepsilon h) = \int_{\mathcal{W}} F(e^{-t}\omega+\beta_t y)\exp\left(\varepsilon\frac{e^{-t}}{\beta_t}\delta h - \frac{\varepsilon^2 e^{-2t}}{\beta_t^2}\|h\|_{\mathcal{H}}^2\right)\ \mathrm{d}\mu(y).$$

Since,

$$\frac{d}{d\varepsilon}\left(\varepsilon\frac{e^{-t}}{\beta_t}\delta h - \frac{\varepsilon^2 e^{-2t}}{\beta_t^2}\|h\|_{\mathcal{H}}^2\right)\Bigg|_{\varepsilon=0} = \frac{e^{-t}}{\beta_t}\delta h,$$

the result follows by dominated convergence.

The main application of the properties of the Ornstein-Uhlenbeck operator are the Meyer inequalities which merely state that an equivalence of norm.

Theorem 3.14 (Meyer inequalities). *For any $p>1$ and any $k\ge 1$, there exist $c_{p,k}$ and $C_{p,k}$ such that for any $F\in\mathbb{D}_{p,k}$,*

$$c_{p,k}\,\mathbf{E}\left[\|\nabla^{(k)}F\|_{\mathcal{H}^{\otimes k}}^p\right]^{1/p} \le \mathbf{E}\left[|(\mathrm{I}+L)^{k/2}F|^p\right] \le C_{p,k}\,\mathbf{E}\left[\|\nabla^{(k)}F\|_{\mathcal{H}^{\otimes k}}^p\right]^{1/p}.$$

3.3 Exercises

Exercise 3.1. Consider the Brownian sheet W which is the centered Gaussian process indexed by $[0,1]^2$ with covariance kernel

$$\mathbf{E}\left[W(t_1,t_2)W(s_1,s_2)\right] = s_1\wedge t_1\ s_2\wedge t_2 := R(s,t).$$

Let $(X_{ij}, 1\le i,j\le N)$ a family of N^2 independent and identically distributed random variables with mean 0 and variance 1. Define

$$S_N(s,t) = \frac{1}{N}\sum_{i=1}^{[Ns]}\sum_{j=1}^{[Nt]} X_{ij}.$$

1. Show that $\big(S_N(s_1,s_2), S_N(t_1,t_2)\big)$ converges to a Gaussian random vector of covariance matrix
$$\Gamma = \begin{pmatrix} R(s,s) & R(s,t) \\ R(s,t) & R(t,t)\end{pmatrix}$$

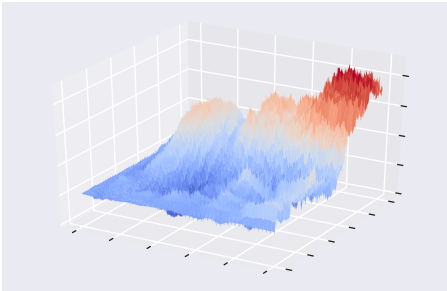

Fig. 3.1 Simulation of a sample of S_N.

2. For $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu)$ and $\omega \in \mathcal{W}$, show that

$$P_t F(\omega) = \mathbf{E}\left[F\big(e^{-t}\omega + W(.,\beta_t^2)\big)\right].$$