CHAPTER 4

POISSON SPACE

4.1 Stochastic geometry

Definition 18. A configuration is a locally finite set of points of a set E: there is a finite number of points in any bounded set. We denote $\mathfrak{N}_E$ as the set of configurations of E.

Example 3 (Bernoulli Process).– The Bernoulli point process is a process based on a finite set $E = \{x_1, \cdots, x_n\}$. Each of these points is ON, independently of others and with probability p. If we introduce $A_1, \cdots, A_n$ random independent variables of Bernoulli distribution with p parameter, we can write:

$$N = \sum_{i=1}^{n} A_i \delta_{x_i}.$$

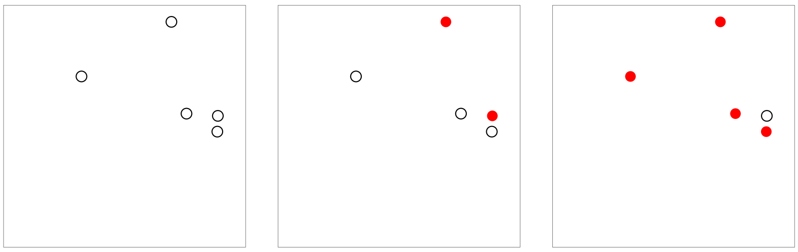


Table 4.1: On the left, the set E. In the middle and at right, two possible realisations. In full (red), the ON points.

Example 4 (Binomial process).– The number of points is fixed to n and μ, a probability measure on $\mathbf{R}^2$ is given. According to μ, the atoms are drawn randomly independent of each other.

We can easily calculate that

$$\mathbf{P}(N(A) = k) = \binom{n}{k} \mu(A)^k (1 - \mu(A))^{n-k}$$

and for the disjoint sets $A_1, \cdots, A_n$

$$\mathbf{P}(N(A_1) = k_1, \cdots, N(A_n) = k_n) = \frac{(k_1 + \ldots + k_n)!}{k_1! \ldots k_n!} \mu(A_1)^{k_1} \ldots \mu(A_n)^{k_n}. \quad (4.1)$$

4.2 Poisson process

The point process, mathematically the richest, is the spatial Poisson process which we recognise as that which generalises the Poisson process on the real line.

Definition 19. Let μ be a Radon measure on a Polish space E that is $\mu(\Lambda) < \infty$ for every compact set $\Lambda \subset E$. The Poisson process with intensity μ is defined by its Laplace transform: for any function $f : E \to \mathbf{R}^+$,

$$\mathbf{E}\left[\exp(-\int f \, \mathrm{d}N)\right] = \exp\left(-\int_E (1 - e^{-f(s)}) \mathrm{d}\mu(s)\right).$$

To clarify that the intensity measure is μ, we will often index the expectation by μ. From the definition of a Poisson process, we immediately infer the Campbell formula by differentiation.

Theorem 44 (Campbell Formula)**.** Let $f \in L^1(E, \mu)$,

$$\mathbf{E}_\mu\left[\int f \, \mathrm{d}N\right] = \int_E f \, \mathrm{d}\mu$$

and if $f \in L^2(E \times E, \mu \otimes \mu)$, then

$$\mathbf{E}_\mu\left[\sum_{x \neq y \in N} f(x, y)\right] = \iint_{E \times E} f(x, y) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y).$$

Particularly, for $f = \mathbf{1}_A$ where A is a compact of E, we note that $\mathbf{E}\left[N(A)\right] = \mu(A)$. If $\mu = \lambda \, \mathrm{d}x$, then λ represents the average number of customers per unit area. An alternative definition is as follows:

Theorem 45. Let μ be a Radon measure on a Polish space E. The Poisson process with intensity μ is the probability measure on $\mathfrak{N}_E$ such that:

- For every compact set $\Lambda \subset E$, $N(\Lambda)$ follows a Poisson distribution with parameter $\mu(\Lambda)$.
- For Λ_1 and Λ_2 two disjoint subsets of $(E, \mathcal{B}(E))$, the random variables $N(\Lambda_1)$ and $N(\Lambda_2)$ are independent.

From this second definition, we immediately deduce the result of the following result of *uniformity*.

Theorem 46. Let N be a Poisson process with intensity μ. Let $\Lambda \subset E$ be a compact set. Given that $N(\Lambda) = n$, the atoms are distributed according to a binomial process for $\mu_\Lambda(A) = \mu(A \cap \Lambda)/\mu(\Lambda)$.

Proof. Let $A_1, \cdots, A_m$ be a partition of Λ or $(k_1, \cdots, k_m)$ such that $k_1 + \ldots + k_m = n$.

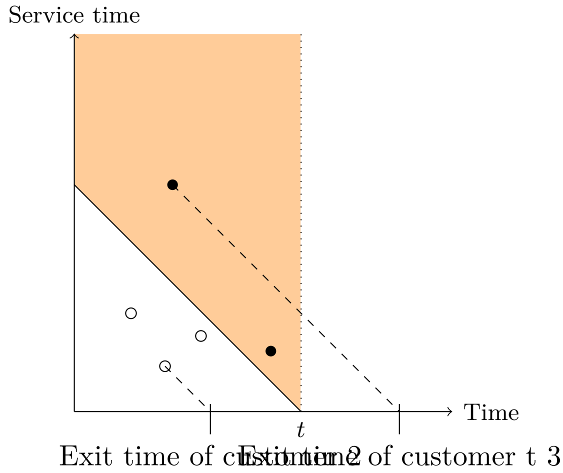
$$\begin{aligned}
\mathbf{P}\Big(N(A_i) = k_i,\, i = 1, \cdots, m \,\Big|\, N(\Lambda) = n\Big) \\
&= \frac{\mathbf{P}(N(A_i) = k_i,\, i = 1, \cdots, m,\, N(\Lambda) = n)}{\mathbf{P}(N(\Lambda) = n)} \\
&= \frac{\mathbf{P}(N(A_i) = k_i,\, i = 1, \cdots, m)}{\mathbf{P}(N(\Lambda) = n)} \\
&= \frac{\exp(-\sum_{i=1}^m \mu(A_i)) \prod_{i=1}^m \frac{\mu(A_i)^{k_i}}{k_i!}}{\exp(-\mu(\Lambda)) \frac{\mu(\Lambda)^n}{n!}} \\
&= \frac{n!}{k_1! \ldots k_m!} \prod_{i=1}^m \left(\frac{\mu(A_i)}{\mu(\Lambda)} \right)^{k_i}.
\end{aligned}$$

According to (4.1) for μ_Λ, we see that, given the number of atoms in Λ, they are distributed according to a binomial process. □

Example 5.– The M/M/∞ queue is the queue with Poisson arrivals, independent and identically distributed from exponential distribution service times, and an infinite number of servers (without buffer). It is initially a theoretical object which is particularly simple to analyse and also a model to which we can compare other situations. Due to the independence of the inter-arrivals and service time, according to the second characterisation of Poisson processes, the process:

$$N = \sum_{n \geq 1} \delta_{(T_n, S_n)}$$

where T_n is the instant of nth arrival and S_n the nth service time, is a Poisson process with $d\mu(t, x) = \lambda \text{ d}t \otimes \mu e^{-\mu x} \text{ d}x$ intensity in $E = \mathbf{R}^+ \times \mathbf{R}^+$.



The customers who are still in service at the time are those who correspond to the points in the shaded trapezium.

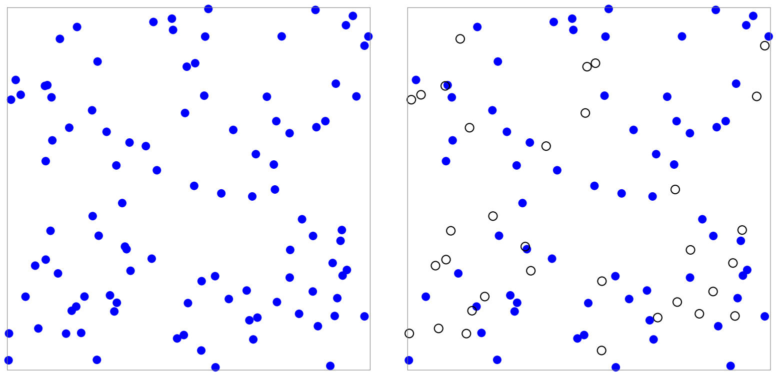


Table 4.2: A realisation of a Poisson process (on the left) and one of its thinning with $p = 2/3$ (on the right). Filled circles correspond to kept points

We deduce that $X(t)$, the number of busy servers at time t follows a Poisson distribution with parameter

$$\int_0^t \left(\int_{t-s}^{\infty} \mu e^{-\mu x} \, \mathrm{d}x \right) \lambda \mathrm{d}s = \lambda \int_0^t e^{-\mu(t-s)} \, \mathrm{d}s = \rho(1 - e^{-\mu t}),$$

where $\rho = \lambda/\mu$. If the system is not empty at time 0, we must add $X(t)$ the number of initial customers still in service at time t. If X_0 follows a Poisson distribution with parameter ρ_0, the number of customers in service at time t follows a Poisson distribution with parameter $\rho_0 e^{-\mu t}$ because each and every customer has a probability $e^{-\mu t}$ of being still in service and the total is thus the thinning of a Poisson random variable. In conclusion, $X(t)$ then follows a Poisson distribution with parameter $\rho + (\rho_0 - \rho)e^{-\mu t}$. Irrespective of the value of ρ_0, the stationary probability of X is a Poisson distribution with parameter ρ.

Theorem 47. Let N^1 and N^2 be two independent Poisson processes with respective intensities μ^1 and μ^2, their superposition N defined by:

$$\int f \, \mathrm{d}N = \int f \, \mathrm{d}N^1 + \int f \, \mathrm{d}N^2 \tag{4.2}$$

is a Poisson process with intensity $\mu^1 + \mu^2$.

Definition 20. Let N be a Poisson process with intensity μ and $p : E \longrightarrow [0, 1]$. The (μ, p)-thinned Poisson process is the process where an atom of the Poisson process N in x is kept with probability $p(x)$.

Theorem 48. A (μ, p)-thinned Poisson process is a Poisson process of intensity μ_p defined by:

$$\mu_p(A) = \int_A p(x) \, \mathrm{d}\mu(x).$$

Theorem [21] is a special case of the displacement theorem.

Definition 21. Let $(\Omega', \mathcal{A}', \mathbf{P}')$ be a probability space and $(F, \mathcal{F})$ a Polish space. A displacement is a measurable application Θ of $\Omega' \times E \longrightarrow F$ such that the random variables $(\Theta(\omega', x), x \in E)$ are independent. For $A \in \mathcal{F}$, we have

$$\theta(x, A) = \mathbf{P}'(\omega' : \Theta(\omega', x) \in A).$$

Thus $\theta(x, A)$ represents the probability that the point x is displaced in A. More mathematically, if we denote by $\Theta(\omega', .)^*\mu$ the image measure of μ through the application $\Theta(\omega', .)$, we have:

$$\mathbf{E}_{\mathbf{P}'}[\Theta^*\mu(A)] = \mathbf{E}_{\mathbf{P}'}\left[\int \mathbf{1}_{\{\Theta(\omega', x)\in A\}} \, \mathrm{d}\mu(x)\right]$$
$$= \int \mathbf{P}'(\Theta(\omega', x) \in A) \, \mathrm{d}\mu(x) = \int \theta(x, A) \, \mathrm{d}\mu(x).$$

This means that:

$$\mathbf{E}_{\mathbf{P}'}\left[\int \mathbf{1}_A \, \mathrm{d}\Theta^*\mu\right] = \int \int_A \theta(x, \, \mathrm{d}y) \, \mathrm{d}\mu(x).$$

Therefore, for a non-negative function f, we obtain:

$$\mathbf{E}_{\mathbf{P}'}\left[\int f \, \mathrm{d}\Theta^*\mu\right] = \int \int f(y)\theta(x, \, \mathrm{d}y) \, \mathrm{d}\mu(x). \tag{4.3}$$

Definition 22. A displacement is said to be conservative when, for any compact $\Lambda \subset E$:

$$\mathbf{E}_{\mathbf{P}'}[\Theta^*\mu(\Lambda)] = \int_\Lambda \int_F \theta(x, \, \mathrm{d}y) \, \mathrm{d}\mu(x) = \mu(A).$$

This means that on average, the *total mass* of the point process is preserved.

Definition 23. Let Θ be a displacement such that $\int_\Lambda \int_F e^{-f(y)}\theta(x, \, \mathrm{d}y) \, \mathrm{d}\mu(x) = \mu(A)$ and N be a point process, the *displaced* point process N^Θ is defined by

$$N^\Theta(\omega') = \sum_{x\in N} \delta_{\Theta(\omega', x)}.$$

Theorem 49. Let N be a Poisson process with intensity μ on E and Θ be a conservative displacement from E to F. The process N^Θ is a Poisson process with intensity μ^Θ defined by

$$\mu^\Theta(A) = \int_E \theta(x, A) \, \mathrm{d}\mu(x).$$

Proof. First, assume that f has a compact support denoted by Λ. We know that given $N(\Lambda)$, the atoms of N are independent, distributed according to $\mu/\mu(\Lambda)$. Therefore, we can write

$$\mathbf{E}\left[F \exp(-\int_\Lambda \, \mathrm{d}N)\right] = \sum_{n=0}^{\infty} \frac{e^{-\mu(\Lambda)}\mu(\Lambda)^n}{n!} \int_{E^n} \prod_{j=1}^{n} e^{-f(x_j)} \frac{\mathrm{d}\mu(x_j)}{\mu(\Lambda)}.$$

According to the construction of N^Θ, the random displacement is independent of N, thus, we have

$$\mathbf{E}\left[\exp(-\int f \mathrm{d}N^\Theta)\right] = \mathbf{E}_{\mathbf{P}'}\left[\sum_{n=0}^\infty \frac{e^{-\mu(\Lambda)}}{n!}\int_{E^n}\prod_{j=1}^n e^{-f(\Theta(\omega',x_j))}\,\mathrm{d}\mu(x_j)\right]$$
$$= \sum_{n=0}^\infty \frac{e^{-\mu(\Lambda)}}{n!}\,\mathbf{E}_{\mathbf{P}'}\left[\int_{E^n}\prod_{j=1}^n e^{-f(\Theta(\omega',x_j))}\,\mathrm{d}\mu(x_j)\right].$$

By definition of a displacement, the random variables $(\Theta(\omega', x_j),\, j = 1, \cdots, n)$ are independent. By using (4.3),we obtain,

$$\mathbf{E}\left[\exp(-\int f\ \mathrm{d}N^\Theta)\right] = \sum_{n=0}^\infty \frac{e^{-\mu(\Lambda)}}{n!}\left(\mathbf{E}_{\mathbf{P}'}\left[\int_E e^{-f(\Theta(\omega',x))}\,\mathrm{d}\mu(x)\right]\right)^n$$
$$= \sum_{n=0}^\infty \frac{e^{-\mu(\Lambda)}}{n!}\left(\int e^{-f}\ \mathrm{d}\mu^\Theta\right)^n$$
$$= \exp\left(-\mu(\Lambda) + \int_\Lambda\int_F e^{-f(y)}\theta(x,\ \mathrm{d}y)\ \mathrm{d}\mu(x)\right).$$

As Θ is conservative, we obtain

$$\mathbf{E}\left[\exp(-\int f\ \mathrm{d}N^\Theta)\right] = \exp\left(-\int_F (1-e^{-f(y)})\int_\Lambda \theta(x,\ \mathrm{d}y)\ \mathrm{d}\mu(x)\right),$$

so N^θ is definitely a Poisson process with intensity μ^Θ.

We obtain the general case for f, by truncation (apply the previous result to $f_\Lambda = f\mathbf{1}_\Lambda$) and by a limit procedure (consider an increasing sequence of compacts $(\Lambda_n,\, n \geq 1)$ such that $\cup_n \Lambda_n = E$. Note that the existence of such a sequence is ensured by the Polish character of E.). □

Proof of Theorem [21.] We consider $F = E \cup \Delta$ where Δ is an external point. With probability $p(x)$, the atom x stays in x, with the complementary probability, it is moved to Δ. This displacement is conservative as we keep the same number of atoms. The restriction at E of the process thus obtained is the thinning of the initial process. Theorem [21] is then a direct consequence of Theorem [22]. □

By applying Theorem [22] to the function $(x \in \mathbf{R}^d \mapsto rx)$ where $r \in \mathbf{R}^+$, we obtain a scaling property which is very useful in many applications.

Corollary 50. Let N be a Poisson process with intensity μ on $\mathbf{R}^d$. Let $r > 0$, N^r is the dilation of N process defined by

$$N^{(r)} = \sum_{x\in N}\delta_{rx}.$$

The process N^r is a Poisson process with intensity $\mu^{(r)}$ where $\mu^{(r)}(A) = \mu(A/r)$ for any $A \in \mathcal{B}(E)$.

Corollary 51. Let N be a Poisson process with intensity $\lambda\,\mathrm{d}x$ on $\mathbf{R}^d$. The process of modules is independent of the process of arguments. The first is a Poisson process with intensity $2\lambda\pi r dr$, and the second is a Poisson process of intensity $(2\pi)^{-1}\mathbf{1}_{[0,\,2\pi]}(\theta)\,\mathrm{d}\theta$.

Proof. Theorem [22] implies that

$$\hat{N} = \sum_{x\in\mathbf{N}} \delta_{\|x\|,\,\mathrm{Arg}(x)}$$

is a Poisson process with $\lambda r \mathbf{1}_{[0,\,2\pi]}(\theta)\,\mathrm{d}r\,\mathrm{d}\theta$ intensity. Hence, we have the result. □

4.3 Stochastic analysis

Theorem 52 (Cameron-Martin theorem)**.** Let N and N' be two point Poisson processes, with respective intensity μ and μ'. Let us assume that $\mu' \ll \mu$ and let us denote $p = \mathrm{d}\mu'/\,\mathrm{d}\mu$. Let Λ be a compact of E. Moreover, if p belongs to $L^1(\mu_\Lambda)$, then for every bounded function F, we have

$$\mathbf{E}\left[F(N'_\Lambda)\right] = \mathbf{E}\left[F(N_\Lambda)\exp\left(\int \ln p\,\mathrm{d}N_\Lambda + \int_\Lambda (1-p)\,\mathrm{d}\mu\right)\right].$$

Proof. We verify this identity for the exponential functions F of the form $\exp(-\int f\,\mathrm{d}N)$ with f at compact support. On the basis of the definition [10],

$$\begin{aligned}
\mathbf{E}&\left[\exp(-\int f\mathrm{d}N_\Lambda)\exp\left(\int_\Lambda \ln p\mathrm{d}N_\Lambda + \int (1-p)\,\mathrm{d}\mu\right)\right] \\
&= \mathbf{E}\left[\exp(-\int (f-\ln p)\,\mathrm{d}N_\Lambda)\right]\exp(\int_\Lambda (1-p)\,\mathrm{d}\mu) \\
&= \exp(-\int (1-\exp(-f+\ln p))\,\mathrm{d}\mu + \int_\Lambda (1-p)\,\mathrm{d}\mu) \\
&= \exp(-\int_\Lambda (1-e^{-f})p\,\mathrm{d}\mu) \\
&= \mathbf{E}\left[F(N'_\Lambda)\right].
\end{aligned}$$

As a result, the measures on $\mathfrak{N}_E$, $\mathbf{P}_{N'_\Lambda}$, and $R\,\mathrm{d}\mathbf{P}_{N_\Lambda}$ where

$$R = \exp\left(\int \ln p\,\mathrm{d}N_\Lambda + \int (1-p)\,\mathrm{d}\mu\right)$$

have the same Laplace transform. Therefore, they are equal and the result follows for any bounded function F . □

In what follows, for a configuration η

$$\eta \oplus x = \begin{cases} \eta, & \text{if } x \in \eta, \\ \eta \cup \{x\}, & \text{if } x \notin \eta. \end{cases}$$

Similarly,

$$\eta \ominus x = \begin{cases} \eta \backslash \{x\}, & \text{if } x \in \eta, \\ \eta, & \text{if } x \notin \eta. \end{cases}$$

As μ is assumed to be diffuse $\mathbf{E}\left[N(\{x\})\right] = \mu(\{x\}) = 0$. Therefore, for fixed x, almost surely, η does not contain x. One of the essential formulas for the Poisson process is the following.

Theorem 53 (Campbell-Mecke formula)**.** Let N be a Poisson process with intensity μ. For any random field $F : \mathfrak{N}_E \times E \to \mathbf{R}$ such that

$$\mathbf{E}\left[\int_E |F(N,\, x)| \, \mathrm{d}\mu(x)\right] < \infty$$

then

$$\mathbf{E}\left[\int_E F(N,\, x) \, \mathrm{d}\mu(x)\right] = \mathbf{E}\left[\int_E F(N \ominus x,\, x) \, \mathrm{d}N(x)\right]. \tag{4.4}$$

Proof. According to the first definition of the Poisson process, for f with compact support and Λ a compact E, for any $t > 0$,

$$\mathbf{E}\left[\exp(-\int (f + t\mathbf{1}_\Lambda) \, \mathrm{d}N)\right] = \exp(-\int_E 1 - e^{-f(x) - t\mathbf{1}_\Lambda(x)} \, \mathrm{d}\mu(x)).$$

According to the theorem of derivation under the summation sign, on one hand, we have

$$\frac{d}{dt}\mathbf{E}\left[\exp(-\int (f + t\mathbf{1}_\Lambda)\mathrm{d}N)\right]\Bigg|_{t=0} = -\mathbf{E}\left[e^{-\int f \, \mathrm{d}N} \int \mathbf{1}_\Lambda \, \mathrm{d}N\right]$$

and on the other hand,

$$\frac{d}{dt}\exp(-\int_E 1 - e^{-f(x) - t\mathbf{1}_\Lambda(x)} \, \mathrm{d}\mu(x))\Bigg|_{t=0} = -\mathbf{E}\left[\int e^{-\int f \, \mathrm{d}N + f(x)} \mathbf{1}_\Lambda(x) \, \mathrm{d}\mu(x)\right].$$

As $\int f \, \mathrm{d}N - f(x) = \int f \, \mathrm{d}(N \ominus x)$, (4.4) is true for functions of the form $\mathbf{1}_\Lambda e^{-\int f \, \mathrm{d}N}$. We admit that this is enough as far as the result is true for all the F functions such that both the members are well defined. □

Definition 24 (Discrete gradient)**.** Let N be a Poisson process with intensity μ. Let $F : \mathfrak{N}_E \longrightarrow \mathbf{R}$ be a measurable function such that $\mathbf{E}\left[F^2\right] < \infty$. We define

$DomD$ as the set of square integrable random variables such that

$$\mathbf{E}\left[\int_E |F(N \oplus x) - F(N)|^2 \, \mathrm{d}\mu(x)\right] < \infty.$$

For $F \in \operatorname{Dom} D$, we set

$$D_x F(N) = F(N \oplus x) - F(N).$$

Example 6.– For example, for f deterministic belonging to $L^2(\mu)$, $F = \int f \, \mathrm{d}N$ belongs to $\operatorname{Dom} D$ and $D_x F = f(x)$ because

$$F(N \oplus x) = \sum_{y \in N \cup \{x\}} f(y) = \sum_{y \in N} f(y) + f(x).$$

Similarly, if $F = \max_{y \in N} f(y)$ then

$$D_x F(N) = \begin{cases} 0 & \text{if } f(x) \leq F(N), \\ f(x) - F & \text{if } f(x) > F(N). \end{cases}$$

Definition 25 (Poisson divergence)**.** We denote by $\operatorname{Dom}_2 \delta$, the set of vector fields such that

$$\mathbf{E}\left[\left(\int_E U(N \ominus x, x)(\, \mathrm{d}N(x) - \, \mathrm{d}\mu(x))\right)^2\right] < \infty.$$

Then, for such vector fields U,

$$\delta U(N) = \int_E U(N \ominus x, x)(\, \mathrm{d}N(x) - \, \mathrm{d}\mu(x)).$$

A consequence of Campbell-Mecke formula is the integration by parts formula.

Theorem 54 (Integration by parts for Poisson process)**.** For $F \in \operatorname{Dom} D$ and any $U \in \operatorname{Dom}_2 \delta$,

$$\mathbf{E}\left[\int_E D_x F(N) \, U(N, x) \, \mathrm{d}\nu(x)\right] = \mathbf{E}\left[F(N) \, \delta U(N)\right].$$

Moreover, we have the analog to (2.14)

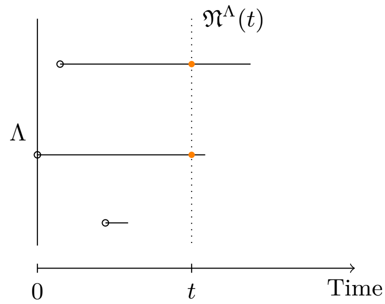
Corollary 55 (Skorohod isometry)**.** For any $U \in \operatorname{Dom}_2 \delta$,

$$\mathbf{E}\left[\delta U^2\right] = \mathbf{E}\left[\int_E U(N, x)^2 \, \mathrm{d}\mu(x)\right] + \mathbf{E}\left[\int_E \int_E D_x U(N, y) \, D_y U(N, x) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y)\right].$$

Let μ be a Radon measure on a Polish space E and Λ be a compact of E. We introduce the Glauber-Poisson process, which is denoted by $\mathfrak{N}^\Lambda$, whose dynamics is as follows:

- $\mathfrak{N}^\Lambda(0) = \eta \in \mathfrak{N}_\Lambda$,
- Each atom of η has a life duration, independent of that of the other atoms, exponentially distributed with parameter 1.
- Atoms are born at moments following a Poisson process with intensity $\mu(\Lambda)$. On its appearance, each atom is localised independently from all the others according to $\mu/\mu(\Lambda)$. It is also assigned in an independent manner, a life duration exponentially distributed with parameter 1.

Figure 4.1: Realisation of a trajectory of $\mathfrak{N}^\Lambda$



At every instant, $\mathfrak{N}^\Lambda(t)$ is a configuration of E. We first observe that the total number of atoms of $\mathfrak{N}^\Lambda(t)$ follows exactly the same dynamics as the number of busy servers in a M/M/∞ queue with parameters $\mu(\Lambda)$ and 1.

Theorem 56 (Glauber process)**.** Assume that $\mathfrak{N}^\Lambda(0)$ is a point Poisson process with intensity ν. At each instant t, the distribution of $\mathfrak{N}^\Lambda(t)$ is that of a Poisson process with intensity $e^{-t}\nu_\Lambda + (1-e^{-t})\mu_\Lambda$ where ν_Λ is the restriction from ν to Λ. Particularly, if $\nu_\Lambda = \mu_\Lambda$, the distribution of $\mathfrak{N}^\Lambda(t)$ does not depend on t and is equal to μ_Λ. We denote $\mathbf{E}_{\mu_\Lambda}[X]$ as the expectation of a random variable X under this induced probability.

Proof. For two disjoint parts A and B of Λ, by construction, the processes $\mathfrak{G}_A$ and $\mathfrak{G}_B$ are independent and follow the same dynamics as that of a M/M/∞ queue with respective parameters $(\mu(A), 1)$ and $(\mu(B), 1)$. The result follows from the properties of the M/M/∞ queue established in example 5. □

As all the sojourn time are exponentially distributed, $\mathfrak{N}^\Lambda$ is a Markov process with values in $\mathfrak{N}_E$. Far from the idea of developing the general theory of Markov processes in the space of measures, we can study its infinitesimal generator and its semi group.

Theorem 57. Let Λ be a compact of E. The infinitesimal generator of $\mathfrak{N}^\Lambda$ is given by

$$-\mathfrak{L}_\Lambda F(N) = \int_\Lambda (F(N \oplus x) - F(N))\, \mathrm{d}\mu(x) + \int (F(N-\delta_x) - F(N))\, \mathrm{d}N(x) \quad (4.5)$$

for F bounded from $\mathfrak{N}_\Lambda$ into $\mathbf{R}$.

Proof. We reason in the same way as that of the Markov process. At a time t, there may be a either a death or a birth. At the time of a departure, we choose the uniformly killed atom among the existing atoms. The death rate is thus $\eta(\Lambda)$ and every atom has a probability $\eta(\Lambda)^{-1}$ of being killed. Therefore, the transition η toward $\eta - \delta_x$ take place at rates of 1 for any $x \in \eta$. The birth rate is $\mu(\Lambda)$ and the position of the new atom is distributed according to the measure $\mu_\Lambda/\mu(\Lambda)$ so the transition η toward $\eta \oplus x$ occurs at a rate $\mathrm{d}\mu_\Lambda(x)$ for each $x \in \Lambda$. From it, we deduce (4.5). □

Theorem 58 (Ergodicity)**.** The semi-group $\mathfrak{P}^\Lambda$ is ergodic. Moreover, $\mathfrak{L}_\Lambda$ is invertible from L^2_0 in L^2_0 where L^2_0 is the subspace of L^2 of the random variables with null expectation and we have

$$\mathfrak{L}_\Lambda^{-1} F = \int_0^\infty \mathfrak{P}_t^\Lambda F\, \mathrm{d}t. \quad (4.6)$$

For any $x \in E$ and any $t > 0$,

$$D_x \mathfrak{P}_t^\Lambda F = e^{-t} \mathfrak{P}_t^\Lambda D_x F. \quad (4.7)$$

In addition,

$$\mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda |D_x(\mathfrak{L}_\Lambda^{-1} F(N))|^2\, \mathrm{d}\mu(x)\right] \le \mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda |D_x F(N)|^2\, \mathrm{d}\mu(x)\right]. \quad (4.8)$$

Proof. Denote $(x_1, \cdots, x_n)$ the atoms of $\mathfrak{N}^\Lambda(0)$ and $(Y_1, \cdots, Y_n)$ some independent random variables exponentially distributed with parameter 1. We set

$$\mathfrak{N}^\Lambda(0)[t] = \sum_{i=1} \mathbf{1}_{\{Y_i \ge t\}} \delta_{x_i},$$

the measure consisting of the atoms of $\mathfrak{N}^\Lambda(0)$ surviving at time t. The distribution of $\mathfrak{N}^\Lambda(t)$ is that of the independent sum of a Poisson process with intensity $(1-e^{-t})\mu_\Lambda$ and of $\mathfrak{N}^\Lambda(0)[t]$. According to Lemma 10.12, we know that for any $F \in L^1$

$$\mathbf{E}_{(1-e^{-t})\mu_\Lambda}[F(N_\Lambda)] = \mathbf{E}_{\mu_\Lambda}\left[F(N_\Lambda)\exp(\ln(1-e^{-t})N(\Lambda) + e^{-t}\mu(\Lambda))\right].$$

Therefore, for any bounded function F and any $\eta \in \mathfrak{N}_\Lambda$, we have the following identity:

$$\begin{aligned}\mathfrak{P}_t^\Lambda F(\eta) &= \mathbf{E}\left[F(\mathfrak{N}^\Lambda(t)) \,|\, \mathfrak{N}^\Lambda(0) = \eta\right] \\ &= \mathbf{E}\left[F(\eta[t] + N_\Lambda)\exp(\ln(1-e^{-t})N(\Lambda) + e^{-t}\mu(\Lambda))\right]. \quad (4.9)\end{aligned}$$

Set

$$R(t) = \exp(\ln(1-e^{-t})N(\Lambda) + e^{-t}\mu(\Lambda)).$$

On the one hand, we have $R(t) \le e^{\mu(\Lambda)}$ and on the other hand, according to definition 10.2, $\mathbf{E}[R(t)] = 1$, and this for any $t \ge 0$. As $\mathfrak{N}^\Lambda(0)$ has a finite number of atoms, $\mathfrak{N}^\Lambda(0)[t]$ almost surely tends toward the zero measure when t tends toward infinity . By dominated convergence, we deduce that

$$\mathfrak{P}_t^\Lambda F(\eta) \xrightarrow{t\to\infty} \mathbf{E}_{\mu_\Lambda}[F(N_\Lambda)]$$

that is to say, $\mathfrak{P}^\Lambda$ is ergodic. The property (4.6) is a well-known relation between the semi-group and infinitesimal generator. Formally, without worrying about the integral convergence, we have

$$\begin{aligned}\mathfrak{L}_\Lambda(\int_0^\infty \mathfrak{P}_t^\Lambda F \mathrm{d}t) &= \int_0^\infty \mathfrak{L}_\Lambda \mathfrak{P}_t^\Lambda F \,\mathrm{d}t \\ &= -\int_0^\infty \frac{d}{dt}\mathfrak{P}_t^\Lambda F \,\mathrm{d}t \\ &= F - \mathbf{E}[F] = F\end{aligned}$$

according to ergodicity of $\mathfrak{P}^\Lambda$ and as F is centered. Let $\mathfrak{N}^\Lambda(t,\, N_\Lambda)$ denote the value of $\mathfrak{N}^\Lambda(t)$ when the initial condition is N_Λ. We can write

$$D_x\mathfrak{P}^\Lambda F(t) = \mathbf{E}\left[\mathfrak{N}^\Lambda(t,\, N_\Lambda \oplus x)\right] - \mathbf{E}\left[\mathfrak{N}^\Lambda(t,\, N_\Lambda)\right].$$

Let Y_x be the life duration of the atom located in x. If $Y_x \ge t$ then the atom is still alive at t, thus $\mathfrak{N}^\Lambda(t,\, N_\Lambda \oplus x) = \mathfrak{N}^\Lambda(t,\, N_\Lambda) \oplus x$. If the atom is already dead at t then $\mathfrak{N}^\Lambda(t,\, N_\Lambda \oplus x) = \mathfrak{N}^\Lambda(t,\, N_\Lambda)$. As Y_x is by construction, independent of N_Λ and $\mathfrak{N}^\Lambda$, it is legitimate to write

$$\begin{aligned}&\mathbf{E}\left[F(\mathfrak{N}^\Lambda(t,\, N_\Lambda \oplus x) \,|\, N_\Lambda\right] - \mathbf{E}\left[F(\mathfrak{N}^\Lambda(t,\, N_\Lambda) \,|\, N_\Lambda\right] \\ &\qquad = \mathbf{E}\left[\mathbf{1}_{\{Y_x \ge t\}}(F(\mathfrak{N}^\Lambda(t,\, N_\Lambda \oplus x)) - F(\mathfrak{N}^\Lambda(t,\, N_\Lambda)) \,|\, N_\Lambda\right] \\ &\qquad\quad + \mathbf{E}\left[\mathbf{1}_{\{Y_x \le t\}}(F(\mathfrak{N}^\Lambda(t,\, N_\Lambda)) - F(\mathfrak{N}^\Lambda(t,\, N_\Lambda))) \,|\, N_\Lambda\right] \\ &\qquad = e^{-t}\mathbf{E}\left[D_x F(\mathfrak{N}^\Lambda(t,\, N_\Lambda))\right]\end{aligned}$$

hence we have the result. According to the representation (4.9) and Jensen's inequality, we see that

$$\left|\mathfrak{P}_t^\Lambda F\right|^2 \le \mathfrak{P}_t^\Lambda F^2. \tag{4.10}$$

Therefore,

$$\begin{aligned}&\int_\Lambda |D_x(\mathfrak{L}_\Lambda^{-1} F(N_\Lambda))|^2 \,\mathrm{d}\mu(x) \\ &\qquad = \int_\Lambda |D_x \int_0^\infty \mathfrak{P}_t^\Lambda F(N_\Lambda) \,\mathrm{d}t|^2 \,\mathrm{d}\mu(x) \\ &\qquad = \int_\Lambda |\int_0^\infty e^{-t}\mathfrak{P}_t^\Lambda D_x F(N_\Lambda) \,\mathrm{d}t|^2 \,\mathrm{d}\mu(x) \\ &\qquad \le \int_\Lambda \int_0^\infty e^{-t}|\mathfrak{P}_t^\Lambda D_x F(N_\Lambda)|^2 \,\mathrm{d}t \,\mathrm{d}\mu(x) \\ &\qquad \le \int_\Lambda \int_0^\infty e^{-t}\mathfrak{P}_t^\Lambda |D_x F(N_\Lambda)|^2 \,\mathrm{d}t \,\mathrm{d}\mu(x) \\ &\qquad = \int_\Lambda \int_0^\infty e^{-t}\mathbf{E}\left[|D_x F|^2(\mathfrak{N}^\Lambda(t)) \,|\, \mathfrak{N}^\Lambda(0) = N_\Lambda\right] \,\mathrm{d}t \,\mathrm{d}\mu(x),\end{aligned}$$

where we have successively used equations (4.6) and (4.7), Jensen's inequality and (4.10). As $\mathfrak{N}^\Lambda(t)$ has the same distribution as $\mathfrak{N}^\Lambda(0)$ if this one is chosen as a Poisson process with μ_Λ intensity, when we take expectations of each side, we obtain the following identity:

$$\begin{aligned}
&\mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda |D_x(\mathfrak{L}_\Lambda^{-1}F(N_\Lambda))|^2 \,\mathrm{d}\mu(x)\right] \\
&\qquad = \mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda \int_0^\infty e^{-t}|D_xF|^2(\mathfrak{N}^\Lambda(t)) \,\mathrm{d}t \,\mathrm{d}\mu(x)\right] \\
&\qquad = \int_\Lambda \int_0^\infty e^{-t}\mathbf{E}_{\mu_\Lambda}\left[|D_xF|^2(N_\Lambda)\right] \,\mathrm{d}t \,\mathrm{d}\mu(x) \\
&\qquad = \mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda |D_xF(N_\Lambda)|^2 \,\mathrm{d}\mu(x)\right].
\end{aligned}$$

Hence, we have the result. □

Theorem 59 (Covariance identity)**.** Let F and G be two functions belonging to $\operatorname{Dom} D$. The following identity is satisfied:

$$\mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda D_xF(N_\Lambda)\, D_xG(N_\Lambda) \,\mathrm{d}\mu(x)\right] = \mathbf{E}_{\mu_\Lambda}\left[F(N_\Lambda)\, \mathfrak{L}_\Lambda G(N_\Lambda)\right].$$

In particular, if G is centered

$$\mathbf{E}_{\mu_\Lambda}\left[F(N_\Lambda)G(N_\Lambda)\right] = \mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda D_xF(N_\Lambda)\, D_x(\mathfrak{L}_\Lambda^{-1}G)(N_\Lambda) \,\mathrm{d}\mu(x)\right]. \tag{4.11}$$

Proof. Let F and G belong to $\operatorname{Dom} D$, according to (4.4) twice and the definition of $\mathfrak{L}_\Lambda$, we have

$$\begin{aligned}
&\mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda D_xF(N_\Lambda)\, D_xG(N_\Lambda) \,\mathrm{d}\mu(x)\right] \\
&\quad = \mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda (F(N_\Lambda) - F(N_\Lambda - \delta_x))(G(N_\Lambda) - G(N_\Lambda - \delta_x)) \,\mathrm{d}N_\Lambda(x)\right] \\
&\quad = \mathbf{E}_{\mu_\Lambda}\left[G(N_\Lambda)\mathfrak{L}_\Lambda F\right] + \mathbf{E}_{\mu_\Lambda}\left[\int G(N_\Lambda)(F(N_\Lambda \oplus x) - F(N_\Lambda)) \,\mathrm{d}\mu(x)\right] \\
&\qquad - \mathbf{E}_{\mu_\Lambda}\left[\int G(N_\Lambda - \delta_x)(F(N_\Lambda) - F(N_\Lambda - \delta_x)) \,\mathrm{d}N_\Lambda(x)\right] \\
&\quad = \mathbf{E}_{\mu_\Lambda}\left[G(N_\Lambda)\mathfrak{L}_\Lambda F(N_\Lambda)\right].
\end{aligned}$$

The result follows. □

Theorem 60. Let N be a Poisson process with intensity μ on E and Λ be a compact of E. Let $F : \mathfrak{N}_\Lambda \to \mathbf{R}$ such that

$$D_xF(N_\Lambda) \le \beta,\ (\mu \otimes \mathbf{P}) - \text{ a.e. and } \int_E |D_xF(N_\Lambda)|^2 \,\mathrm{d}\mu(x) \le \alpha^2,\ \mathbf{P} - a.e..$$

For any $r > 0$, we have the following inequality

$$\mathbf{P}(F(N_\Lambda) - \mathbf{E}\left[F(N_\Lambda)\right] > r) \leq \exp\left(-\frac{r}{2\beta}\ln(1+\frac{r\beta}{\alpha^2})\right).$$

Proof. Let Λ be a compact of E, a bounded function F of zero expectation. According to Theorem 10.17, we can write the following identities:

$$\begin{aligned}\mathbf{E}_{\mu_\Lambda}\left[F(N_\Lambda)e^{\theta F(N_\Lambda)}\right] &= \mathbf{E}_{\mu_\Lambda}\left[\int D_x(\mathfrak{L}_\Lambda^{-1}F(N_\Lambda))\, D_x(e^{\theta F(N_\Lambda)})\, \mathrm{d}\mu(x)\right] \\ &= \mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda D_x(\mathfrak{L}_\Lambda^{-1}F(N_\Lambda))(e^{\theta D_x F(N_\Lambda)}-1)e^{\theta F(N_\Lambda)}\, \mathrm{d}\mu(x)\right].\end{aligned}$$

The function $(x \mapsto (e^x - 1)/x)$ is continuously increasing on $\mathbf{R}$; therefore, we have

$$\begin{aligned}&\mathbf{E}_{\mu_\Lambda}\left[F(N_\Lambda)e^{\theta F(N_\Lambda)}\right] \\ &\quad= \theta\, \mathbf{E}_{\mu_\Lambda}\left[\int_\Lambda D_x(\mathfrak{L}_\Lambda^{-1}F(N_\Lambda))\, D_xF(N_\Lambda)\, \frac{e^{\theta D_xF(N_\Lambda)}-1}{\theta D_xF(N_\Lambda)}\, e^{\theta F(N_\Lambda)}\, \mathrm{d}\mu(x)\right] \\ &\quad\leq \theta\alpha^2\, \frac{e^{\theta\beta}-1}{\theta\beta}\, \mathbf{E}_{\mu_\Lambda}\left[e^{\theta F(N_\Lambda)}\right].\end{aligned}$$

This implies that

$$\frac{d}{d\theta}\log \mathbf{E}_{\mu_\Lambda}\left[e^{\theta F(N_\Lambda)}\right] \leq \alpha^2\frac{e^{\theta\beta}-1}{\beta}.$$

Therefore,

$$\mathbf{E}_{\mu_\Lambda}\left[e^{\theta F(N_\Lambda)}\right] \leq \exp\left(\frac{\alpha^2}{\beta}\int_0^\theta (e^{\beta u}-1)\, \mathrm{d}u\right).$$

For $x > 0$, for any $\theta > 0$,

$$\begin{aligned}\mathbf{P}(F(N_\Lambda) > x) &= \mathbf{P}(e^{\theta F(N_\Lambda)} > e^{\theta x}) \\ &\leq e^{-\theta x}\mathbf{E}\left[e^{\theta F(N_\Lambda)}\right] \leq e^{-\theta x}\exp\left(\frac{\alpha^2}{\beta}\int_0^\theta (e^{\beta u}-1)\mathrm{d}u\right). \qquad (4.12)\end{aligned}$$

This result is true for any θ, so we can optimise with respect to θ. At fixed x , we search the value of θ which cancels the derivative of the right-hand-side with respect to θ. Plugging this value into (4.12), we can obtain the result. □