Chapter 2

Wiener space

§ 1 Gaussian random variables

We begin by basic definitions about Gaussian random variables and vectors.

Definition 2.1 (Gaussian random variable)**.** A real valued random variable X is Gaussian whenever its characteristic function is of the form

$$\mathbf{E}\left[e^{itX}\right] = e^{itm}e^{-\sigma^2 t^2/2}.$$

It is well known that $\mathbf{E}[X] = m$ and $\operatorname{Var}(X) = \sigma^2$.

This definition means that whenever we know that a random variable is Gaussian, it is sufficient to compute its average and its variance to fully determine its distribution. A Gaussian random vector is not simply a collection of Gaussian random variables. It is true that all the coordinates of a Gaussian vector are Gaussian but they do satisfy a supplementary condition. In what follows, the Euclidean scalar product on $\mathbf{R}^n$ is defined by

$$\langle x,\, y\rangle = \sum_{j=1}^{n} x_j y_j.$$

Definition 2.2 (Gaussian random vector)**.** A random vector X in $\mathbf{R}^n$, i.e. $X = (X_1, \cdots, X_n)$, is a Gaussian random vector whenever for any $t = (t_1, \cdots, t_n) \in \mathbf{R}^n$, the real-valued random variable

$$\langle t,\, X\rangle = \sum_{j=1}^{n} t_j X_j$$

is Gaussian.

In view of the remark following the definition 2.1, we have

$$\mathbf{E}\left[e^{i\langle t,\, X\rangle}\right] = e^{i\langle t,\, m\rangle} e^{-\frac{1}{2}\langle \Gamma_X t,\, t\rangle}, \tag{2.1}$$

where

$$\Gamma_X = \Big(\operatorname{cov}(X_j, X_k),\ 1 \le j, k \le n\Big).$$

is the so-called covariance matrix of X.

Remark.– Somehow hidden in the previous definition lies the identity

$$\operatorname{Var}\langle t,\, X\rangle = \sum_{i,j=1}^{n} \operatorname{cov}(X_j, X_k) t_i t_j \tag{2.2}$$

for any $t = (t_1, \cdots, t_n) \in \mathbf{R}^n$. Since a variance is always non-negative, this means that Γ_X satisfies the identity

$$\langle \Gamma_X t,\, t\rangle = \sum_{i,j=1}^{n} \Gamma_X(i,j)\, t_i t_j \geq 0,$$

which induces that the eigenvalues of Γ_X are non-negative.

The main feature of Gaussian vectors is that they are stable by affine transformation.

Theorem 2.1. Let X be an $\mathbf{R}^n$-valued Gaussian vector, $B \in \mathbf{R}^p$ and A a linear map (i.e. a matrix) from $\mathbf{R}^n$ into $\mathbf{R}^p$. The random $Y = AX + B$ is an $\mathbf{R}^p$-valued Gaussian vector whose characteristics are given by

$$\mathbf{E}\left[Y\right] = A\mathbf{E}\left[X\right] + B,\ \Gamma_Y = A\Gamma_X A^t,$$

where A^t is the transpose of A.

Remark.– If Γ is non-negative symmetric matrix, one can define $\Gamma^{1/2}$, a symmetric non-negative matrix whose square equals Γ. If $X = (X_1, \cdots, X_n)$ is a vector of independent standard Gaussian random variables, then the previous theorem entails that $\Gamma^{1/2}X$ is a Gaussian vector of covariance matrix Γ.

Beyond this stability by affine transformation, the set of Gaussian vectors enjoys another remarkable stability property.

Theorem 2.2. Let $(X_n, n \geq 1)$ be a sequence of Gaussian vectors which converges in distribution to some random vector X. Then, X is a Gaussian vector and the $(\Gamma_{X_n},\, n \geq 1)$ tends to Γ_X.

Remark that for $X \sim \mathcal{N}(0, \mathrm{I}_n)$, a standard Gaussian vector in $\mathbf{R}^n$,

$$\mathbf{E}\left[\|X\|^2_{\mathbf{R}^n}\right] = \sum_{j=1}^{n} \mathbf{E}\left[X_j^2\right] = n.$$

This means that the mean norm of such a random variable goes to infinity as the dimension grows. Thus, we cannot construct a Gaussian distribution on an infinite dimensional space like $\mathbf{R}^{\mathbf{N}}$, by just extending what we do on $\mathbf{R}^n$.

The construction of measures on functional spaces is a delicate question which is satisfactory solved (only) for Gaussian measures. Recall that a Brownian motion is defined as follows.

Definition 2.3. The Brownian motion $B = (B(t),\, t \geq 0)$ is the (unique) centered, Gaussian process on $\mathbf{R}^+$ with independent increments such that

$$\mathbf{E}\left[B(t)B(s)\right] = t \wedge s.$$

Its sample-paths are Hölder continuous of any order strictly less than 1/2.

There are several possibilities to prove the existence of such a process. The most intuitive is probably the Donsker-Lamperti theorem [Don51, Lam62].

Theorem 2.3 (Donsker-Lamperti)**.** Let $(X_n,\, n \geq 1)$ be a sequence of independent, identically random variables such that $\mathbf{E}\left[|X_1|^{2p}\right] < \infty$. Then,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} X_j \Longrightarrow B(t)$$

in the topology of $\operatorname{Hol}(\gamma)$ for any $\gamma < (p-1)/2p$, i.e.

$$\mathbf{E}\left[F\Big(\frac{1}{\sqrt{n}} \sum_{j=1}^{[n.]} X_j\Big)\right] \xrightarrow{n\to\infty} \mathbf{E}\left[F(B)\right]$$

for any $F : \operatorname{Hol}(\gamma) \to \mathbf{R}$ bounded and continuous.
For $p = 1$, i.e. square integrable random variables, the convergence holds in $\mathcal{C}([0,T];\mathbf{R})$ for any $T > 0$.

This means that the distribution of this process is a probability on either $\mathcal{C}$ or $\operatorname{Hol}(\gamma)$ for any $\gamma < 1/2$. However, the construction of the Brownian motion via the random walk is not fully satisfactory as we cannot write B as the sum of a series. The construction of Itô-Nisio is more interesting in this respect. We need to introduce a few functional spaces before going further. In the sequel, we shall consider different families of fractional Sobolev spaces.

Definition 2.4 (Riemann-Liouville fractional spaces)**.** For $\alpha > 0$, for $f \in L^2([0,1])$,

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, \mathrm{d}s. \tag{2.3}$$

The space $I_{\alpha,2}$ is the set $I^\alpha(L^2[0,1])$ equipped with the scalar product

$$\langle I^\alpha f,\, I^\alpha g\rangle_{I_{\alpha,2}} = \langle f,\, g\rangle_{L^2} = \int_0^1 f(s)g(s)\, \mathrm{d}s.$$

Since the map $(f \mapsto I^\alpha f)$ is one-to-one (cf. [SKM93]), this defines a scalar product.

More generally, for $p \geq 1$, $I_{\alpha,1}$ is the space $I^\alpha(L^p)$ equipped with norm

$$\|I^\alpha f\|_{I_{\alpha,p}} = \|f\|_{L^p}.$$

Another useful scale of functions is the Slobodetzky family of fractional Sobolev spaces.

Definition 2.5 (Slobodetzky spaces)**.** For $\alpha \in (0,1]$ and $p \geq 1$, the space $_{\alpha,p}$ is the space of measurable functions over $[0,1]$ such that

$$\|f\|_{\alpha,p}^p := \|f\|_{L^p}^p + \iint_{[0,1]^2} \frac{|f(t)-f(s)|^p}{|t-s|^{1+\alpha p}} \, \mathrm{d}s \, \mathrm{d}t < \infty.$$

These spaces are interesting essentially because of the following embeddings [SKM93].

Theorem 2.4. For any $\alpha'' > \alpha' > \alpha > 1/p$, we have

$$I_{\alpha'',p} \subset W_{\alpha',p} \subset I_{\alpha,p} \subset \mathrm{Hol}(\alpha - 1/p) \subset \mathcal{C}.$$

Moreover, since polynomials belong to any $I_{\alpha,p}$ and are dense in $\mathcal{C}$, the space of continuous functions on $[0,1]$, all these spaces are dense in $\mathcal{C}$.

As a consequence, we retrieve easily the Kolmogorov lemma about the regularity of Brownian sample-paths.

Lemma 2.5. For any $\alpha \in [0,1/2)$ and any $p \geq 1$, the sample-paths of a Brownian motion belong to $\mathrm{W}_{\alpha,p}$ with probability 1.

Proof. It is sufficient to prove that

$$\mathbf{E}\left[\iint_{[0,1]^2} \frac{|B(t)-B(s)|^p}{|t-s|^{1+\alpha p}} \, \mathrm{d}s \, \mathrm{d}t\right] < \infty.$$

Since $B(t)-B(s)$ is a Gaussian random variable,

$$\mathbf{E}\left[|B(t)-B(s)|^p\right] = c_p \, \mathbf{E}\left[|B(t)-B(s)|^2\right]^{p/2} = c_p |t-s|^{p/2}.$$

The function $(s,t) \longmapsto |t-s|^{-1+(1/2-\alpha)p}$ is integrable provided that $\alpha < 1/2$, hence the result. □

An alternative construction is due to Itô and Nisio [IN68]. Consider $(\dot{h}_m, \, m \geq 0)$ a complete orthonormal basis of L^2. By the very definition of the scalar product on $\mathcal{H}$, this entails that $(h_m = I^1 \dot{h}_m, \, m \geq 0)$ is a complete orthonormal basis of $\mathcal{H}$. One may choose the family given by:

$$h_0(t) = t \text{ and } h_m(t) = \frac{\sqrt{2}}{\pi m} \sin(\pi m t) \text{ for } m \geq 1.$$

Then, consider the sequence of approximations given by

$$S_n(t) = \sum_{m=0}^{n} X_m\, h_m(t) \tag{2.4}$$

where $(X_m,\, m \geq 0)$ is a sequence of independent, standard Gaussian random variables. We then have the following extension of the Itô-Nisio theorem.

Theorem 2.6. For any (α, p) such that $0 < \alpha - 1/p < 1/2$, the sequence $(S_n, n \geq 1)$ converges in $_{\alpha,p}$ with probability 1. Moreover, the limit process, denoted by B, is Gaussian, centered with covariance

$$\mathbf{E}\left[B(t)B(s)\right] = \min(t, s).$$

Hence B is distributed as a Brownian motion.

We first need a general lemma.

Lemma 2.7. Let

$$\omega_M = \sum_{m,n \geq M} \|S_n - S_m\|_{W_{\eta,p}} \text{ and } T_M = \sup_{n \geq M} \|S_n - S_M\|_{W_{\eta,p}}.$$

If $(T_M,\, M \geq 1)$ converges in probability to 0 then $(S_n,\, n \geq 1)$ is convergent with probability 1.

Proof. It is clear that

$$\Big(T_M \leq \epsilon\Big) \Longrightarrow \Big(\omega_M \leq 2\epsilon\Big),$$

hence

$$\mathbf{P}(\omega_M > 2\epsilon) \leq \mathbf{P}(T_M > \epsilon).$$

If $(T_M,\, M \geq 1)$ converges in probability to 0, then so does $(\omega_M,\, M \geq 1)$. Consequently, there is a subsequence which converges with probability 1 but ω_M is decreasing, hence the whole sequence $(\omega_M,\, M \geq 1)$ converges to 0 with probability 1.

This means that $(S_n,\, n \geq 1)$ is a.e. a Cauchy sequence in a complete Banach space, hence is convergent. □

Proof of Theorem 2.6. The Doob inequality for Banach valued martingales states that

$$\mathbf{E}\left[T_M^p\right] \leq \frac{p}{p-1} \sup_{n \geq M} \mathbf{E}\left[\|S_n - S_M\|_{W_{\eta,p}}^p\right]. \tag{2.5}$$

Since $S_n - S_M$ is a Gaussian process

$$\begin{aligned}\mathbf{E}\left[\left|(S_n-S_M)(t)-(S_n-S_M)(s)\right|^p\right] \\ &= c_p \mathbf{E}\left[\left|(S_n-S_M)(t)-(S_n-S_M)(s)\right|^2\right]^{p/2} \\ &= c_p \mathbf{E}\left[\left(\sum_{m=M+1}^{n} X_m\left(h_m(t)-h_m(s)\right)\right)^2\right]^{p/2}.\end{aligned}$$

Since the X_m's are independent with unit variance,

$$\mathbf{E}\left[\left(\sum_{m=M+1}^{n} X_m\left(h_m(t)-h_m(s)\right)\right)^2\right] = \sum_{m=M+1}^{n}\left(h_m(t)-h_m(s)\right)^2. \tag{2.6}$$

The trick is to note that

$$h_m(t) = \langle \dot{h}_m, \mathbf{1}_{[0,t]}\rangle_{L^2} = \langle h_m,\, t\wedge .\rangle_{\mathcal{H}}.$$

This means that the right-hand-side of (2.6) is the Cauchy remainder of the series

$$\sum_{m=0}^{\infty}\langle h_m,\, t\wedge . - s\wedge .\rangle_{\mathcal{H}}^2 = \|t\wedge . - s\wedge .\|_{\mathcal{H}}^2 = |t-s|,$$

according to the Parseval identity. Since $\eta < 1/2$,

$$\int_{[0,1]^2} |t-s|^{p/2}|t-s|^{-1-\eta p}\,\mathrm{d}s\,\mathrm{d}t = \int_{[0,1}^2 |t-s|^{-1+(1/2-\eta)p}\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

It follows that

$$\begin{aligned}\sup_{n\ge M}\mathbf{E}\left[\|S_n-S_M\|_{W_{\eta,p}}^p\right] \\ &\le c_p\int_{[0,1]}^2\left(\sum_{m=M+1}^{\infty}\langle h_m,\, t\wedge . - s\wedge .\rangle_{\mathcal{H}}^2\right)^{p/2}|t-s|^{-1-\eta p}\,\mathrm{d}s\,\mathrm{d}t \xrightarrow{M\to\infty} 0,\end{aligned}$$

by the dominated convergence theorem. The result follows from (2.5), the Markov inequality and Lemma 2.7.

We mentioned in Theorem 2.2 that the limit of Gaussian vectors are automatically Gaussian, the same holds similarly for Gaussian processes hence the limit of S_n is a Gaussian process, which we denote by B.

We have

$$\begin{aligned}\mathbf{E}\left[B(t)B(s)\right] &= \mathbf{E}\left[\sum_{m=0}^{\infty} X_m\langle h_m,\, t\wedge .\rangle_{\mathcal{H}} \times \sum_{m'=0}^{\infty} X_{m'}\langle h_{m'},\, s\wedge .\rangle_{\mathcal{H}}\right] \\ &= \mathbf{E}\left[\sum_{m=0}^{\infty} X_m^2\langle h_m,\, t\wedge .\rangle_{\mathcal{H}}\langle h_m,\, s\wedge .\rangle_{\mathcal{H}}\right] \\ &= \sum_{m=0}^{\infty}\langle h_m,\, t\wedge .\rangle_{\mathcal{H}}\langle h_m,\, s\wedge .\rangle_{\mathcal{H}} \\ &= \langle t\wedge .,\, s\wedge .\rangle_{\mathcal{H}},\end{aligned}$$

according to the Parseval equality. The very definition of the scalar product on $\mathcal{H}$ entails that

$$\langle t \wedge ., \, s \wedge . \rangle_{\mathcal{H}} = \int_0^1 \mathbf{1}_{[0,t]}(r)\mathbf{1}_{[0,s]}(r) \, \mathrm{d}r = t \wedge s.$$

□

Several other constructions as limit of stochastic processes lead to a Brownian motion. As a conclusion of these theorems, it appears that the distribution of B is a probability measure on the Banach spaces $\mathcal{C}([0,1];\mathbf{R})$, $\mathrm{Hol}(\gamma)$ or $\mathrm{W}_{\alpha,p}$. Now, if we reverse the problem, how can we characterize a probability measure on, say, $\mathcal{C}([0,1];\mathbf{R})$? How do we determine that it coincides with the Brownian motion distribution?

In finite dimension, a probability measure is characterized by its Fourier transform, often called its characteristic function. As shown in [IN68], this still holds in separable Banach spaces.

Definition 2.6. For μ a probability measure on a separable Banach space (whose dual is denoted by *), its characteristic functional is

$$\begin{aligned} \phi_\mu \, : \, ^* &\longrightarrow \mathbf{C} \\ z &\longmapsto \int e^{i\langle z, w\rangle_{*,}} \, \mathrm{d}\mu(w). \end{aligned}$$

Theorem 2.8. For μ and ν two probability measures on W,

$$(\phi_\mu = \phi_\nu) \Longrightarrow (\mu = \nu).$$

From now on, will be any of the space $\mathrm{W}_{\alpha,p}$ for $0 < \alpha - 1/p < 1/2$ or $\mathcal{C}([0,1],\mathbf{R})$ and * is its dual. In preparation of the chapter about fractional Brownian motion, we reserve the notation $\mathcal{H}$ to the space $I_{1,2}$.

The Hilbert space $I_{1,2}$ is the Riemann-Liouville fractional space of functions with a square integrable derivative. It plays the rôle of pivotal space, meaning that it is identified it with its dual. This is represented by the $\simeq$ symbol. The map emb is the embedding from $I_{1,2}$ into and emb^* is its adjoint map: For any $z \in^*$ and $h \in I_{1,2}$,

$$\langle z, \mathrm{emb}(h)\rangle_{*,} = \langle \mathrm{emb}^*(z), h\rangle_{I_{1,2}}.$$

Example 1.– As the map I^1, the first order quadrature operator, is an isometric isomorphism between L^2 and $\mathcal{H}$, it is common to identify these two spaces. Since we already identified $\mathcal{H}$ and its dual, it must not be done without great care.

According to Theorem 2.4, $I_{1,2} \subset \mathrm{Hol}(1/2)$. Thus, the Dirac measure ε_a is a continuous linear map on $I_{1,2}$. Let x_a be its representation in $I_{1,2}$. We must have for any $f \in I_{1,2}$,

$$\varepsilon_a(f) = f(a) = f(a) - f(0) = \langle x_a, f\rangle_{I_{1,2}} = \int_0^1 \dot{x_a}(s)\dot{f}(s) \, \mathrm{d}s,$$

where $\dot{f} = (I^1)^{-1} f$ is the derivative of f. The sole candidate is $\dot{x}_a = \mathbf{1}_{[0,a]}$, hence $x_a(s) = a \wedge s$, i.e. $\mathrm{emb}^*(\varepsilon_a) = . \wedge a$.

Hence, $\mathrm{emb}^*(\varepsilon_a) = a \wedge .$, which corresponds to the function $\mathbf{1}_{[0,a]}$ in L^2.

For the sequel, it is useful to have in mind the diagram 2.1.

$$\begin{array}{ccc} {}^* & \xrightarrow{\mathrm{emb}^*} & \mathcal{H}^* = (I_{1,2})^* \\ & & \downarrow \simeq \\ L^2 & \xrightarrow{I^1 \quad \mathrm{emb}^r} & \mathcal{H} = I_{1,2} \end{array}$$

Figure 2.1: Embeddings and identification for Wiener spaces.

With the notations of Theorem 2.6, we have

Theorem 2.9. For any $z \in^*$,

$$\mathbf{E}\left[e^{i\langle z, B\rangle_{*,}}\right] = \exp\left(-\frac{1}{2}\| \mathrm{emb}^*(z)\|_{\mathcal{H}}^2\right). \tag{2.7}$$

Proof. From Theorem 2.6, we have

$$\langle z, B\rangle_{*,} = \lim_{n\to\infty} \sum_{m=0}^{n} X_m \left\langle z, \mathrm{emb}(h_m)\right\rangle_{*,}.$$

Remark that the random variable $\langle z, B\rangle_{*,}$ is the limit of a sum of independent Gaussian random variables. By dominated convergence, we get

$$\begin{aligned} \mathbf{E}\left[e^{i\langle z, B\rangle_{*,}}\right] &= \lim_{n\to\infty} \prod_{m=0}^{n} \mathbf{E}\left[iX_m \left\langle z, \mathrm{emb}(h_m)\right\rangle_{*,}\right] \\ &= \lim_{n\to\infty} \prod_{m=0}^{n} \exp\left(-\frac{1}{2}\langle z, \mathrm{emb}(h_m)\rangle_{*,}^2\right) \\ &= \exp\left(-\frac{1}{2}\sum_{m=0}^{\infty}\langle \mathrm{emb}^*(z),\, h_m\rangle_{*,}^2\right) \\ &= \exp\left(-\frac{1}{2}\| \mathrm{emb}^*(z)\|_{\mathcal{H}}^2\right), \end{aligned}$$

according to the Parseval equality. □

The dual bracket between an element of * and an element of is defined by construction of the dual of . But we not only have the Banach structure on , we also have a measure. We can take advantage of this richer framework to extend the above mentioned dual bracket to elements of $\mathcal{H}$ and .

In what follows, the letter ω represents the generic element of . We denote by μ the distribution of B on .

Definition 2.7 (Wiener integral)**.** The map

$$\begin{aligned} \delta \,:\, \operatorname{emb}^*(^*) \subseteq \mathcal{H} &\longrightarrow L^2(\mu) \\ \operatorname{emb}^*(z) &\longmapsto \langle z,\, \omega\rangle_{*,}\,. \end{aligned}$$

is an isometry. Its unique extension to $\mathcal{H}$ is called the Wiener integral.

Proof. The very definition of μ entails that for any $z \in^*$, the random variable $(\delta z)(\omega) = \langle z,\, \omega\rangle_{*,}$ is a centered Gaussian random variable of variance $\|\operatorname{emb}^*(z)\|^2_{\mathcal{H}}$. Otherwise stated, for $h \in \operatorname{emb}^*(W^*)$,

$$\|\delta(h)\|_{L^\mu} = \|h\|_{\mathcal{H}}.$$

Since $\operatorname{emb}(\mathcal{H})$ is dense in , then $\operatorname{emb}^*(^*)$ is dense in $\mathcal{H}$. Thus, we can extend δ as a linear isometry from $\mathcal{H}$ into $L^2(\mu)$. □

Remark.– For $h \in^*$ and $k \in \mathcal{H}$

$$\langle h, \omega + \operatorname{emb}(k)\rangle_{*,} = \delta(\operatorname{emb}^*(h))(\omega) + \langle \operatorname{emb}^*(h),\, k\rangle_{\mathcal{H}}\,.$$

Passing to the limit, we have

$$\delta h(\omega + k) = \delta h(\omega) + \langle h,\, k\rangle_{\mathcal{H}}\,, \tag{2.8}$$

for any $h \in \mathcal{H}$.

Remark.– As we have seen above, $\|s \wedge .\|_{\mathcal{H}} = s$. Moreover, for any $0 \le s_1 < \ldots < s_n \le 1$,

$$\langle s_{i+1} \wedge . - s_i \wedge .,\; s_{j+1} \wedge . - s_j \wedge .\rangle_{\mathcal{H}} = (s_{i+1} - s_i)\mathbf{1}_{\{i\}}(j).$$

Hence, the vector

$$\Big(\delta(s_{i+1} \wedge . - s_i \wedge .),\, i = 0, \cdots, n-1\Big)$$

has the distribution of the corresponding increments of a Brownian motion.

Remark.– Furthermore, let $(h_m,\, m \ge 0)$ be a complete orthonormal basis of $\mathcal{H}$. The sequence $(\delta h_m(\omega),\, m \ge 0)$ is a sequence of independent standard Gaussian variables, hence

$$B = \sum_{m=0}^{\infty} \delta h_m\, h_m \tag{2.9}$$

is convergent with probability 1 in and defines a Brownian motion on .

The most useful theorem for the sequel states that if we translate the Brownian sample-path by an element of $\mathcal{H}$, then the distribution of this new process is absolutely continuous with respect to the initial Wiener measure. This is the transposition in infinite dimension of the trivial result in dimension 1:

$$\mathbf{E}\left[F(\mathcal{N}(m,1))\right] = (2\pi)^{-1/2}\int_{\mathbf{R}} f(x+m)e^{-x^2/2}\,\mathrm{d}x \\ = (2\pi)^{-1/2}\int_{\mathbf{R}} f(x)e^{xm-m^2/2}e^{-x^2/2}\,\mathrm{d}x.$$

Theorem 2.10 (Cameron-Martin)**.** For any $h \in \mathcal{H}$, for any bounded function $F :\to \mathbf{R}$,

$$\mathbf{E}\left[F(B+\operatorname{emb}(h))\right] = \mathbf{E}\left[F(B)\exp\Big(\delta h(B) - \frac{1}{2}\|h\|_{\mathcal{H}}^2\Big)\right]. \tag{2.10}$$

Proof. Let

$$\begin{aligned} T_h \,:\, &\longrightarrow \\ \omega &\longmapsto \omega + \operatorname{emb}(h) \end{aligned}$$

whose inverse is T_{-h}. Eqn. (2.10) can be rewritten

$$\mathbf{E}\left[F\circ T_h\right] = \mathbf{E}\left[F\Lambda_h\right] \text{ where } \Lambda_h = \exp\Big(\delta h - \frac{1}{2}\|h\|_{\mathcal{H}}^2\Big).$$

It is equivalent to

$$\mathbf{E}\left[F\circ T_{-h}\ \Lambda_h\right] = \mathbf{E}\left[F\right]. \tag{2.11}$$

This means that the pushforward of the measure $\Lambda_h\mu$ by the map T_{-h} is the Wiener measure μ. In view of (2.7), we have to prove that for any $z \in^*$,

$$\mathbf{E}\left[\exp\Big(i\left\langle z, B-\operatorname{emb}(h)\right\rangle_{*,}\Big)\exp\Big(\delta h(B) - \frac{1}{2}\|h\|_{\mathcal{H}}^2\Big)\right] \\ = \exp\Big(-\frac{1}{2}\|\operatorname{emb}^*(z)\|_{\mathcal{H}}^2\Big). \tag{2.12}$$

Remark that

$$i\left\langle z, B-\operatorname{emb}(h)\right\rangle_{*,} + \delta h(B) - \frac{1}{2}\|h\|_{\mathcal{H}}^2 \\ = i\delta\Big(\operatorname{emb}^*(z) - ih\Big)(B) - i\left\langle \operatorname{emb}^*(z),\, h\right\rangle_{\mathcal{H}} - \frac{1}{2}\|h\|_{\mathcal{H}}^2. \tag{2.13}$$

In view of the definition of the Wiener integral,

$$\mathbf{E}\left[\exp\Big(i\delta\Big(\operatorname{emb}^*(z)-ih\Big)(B)\Big)\right] = \exp\Big(-\frac{1}{2}\|\operatorname{emb}^*(z)-ih\|_{\mathcal{H}}^2\Big).$$

Since $\mathcal{H}$ is a real (not a complex) Hilbert space,

$$\|\operatorname{emb}^*(z)-ih\|_{\mathcal{H}}^2 = \|\operatorname{emb}^*(z)\|_{\mathcal{H}}^2 - \|h\|_{\mathcal{H}}^2 - 2i\left\langle \operatorname{emb}^*(z),\, h\right\rangle_{\mathcal{H}}. \tag{2.14}$$

Plug (2.14) into (2.13) to get (2.12). □

A quick refresher about Hilbert spaces

We shall often encounter partial functions: For a function of several variables, say $f(t, s)$, we denote by $f(t, .)$ the partial function

$$\begin{aligned} f(t,.)\,:\, E &\longrightarrow \mathbf{R} \\ s &\longmapsto f(t,s). \end{aligned}$$

Definition 2.8. A Hilbert space $(H, \langle ., . \rangle_H)$ is a vector space H which is complete for the topology induced by the scalar product $\langle ., . \rangle_H$.

Recall that a metric space E is separable whenever there exists a denumerable family which is dense: There exists $(x_n, n \geq 1)$ such that for any $\epsilon > 0$, any $x \in X$, one can find some x_n such that $d(x, x_n) < \epsilon$. By construction, the set of rational numbers is such a set in $\mathbf{R}$. All the spaces we are going to consider, even the seemingly ugliest, are separable hence we can safely forget this subtlety.

Theorem 2.11. Any separable Hilbert space H admits a complete orthonormal basis (CONB for short) $(e_n,\, n \geq 1)$, i.e. on the one hand

$$\langle e_n,\, e_m \rangle_H = \mathbf{1}_{\{n\}}(m)$$

and on the other hand, any $x \in H$ can be written

$$x = \sum_{n=1}^{\infty} \langle x,\, e_n \rangle_H\, e_n$$

which means

$$\lim_{N\to\infty} \Big\| x - \sum_{n=1}^{N} \langle x,\, e_n \rangle_H\, e_n \Big\|_H = 0.$$

We will use repeatedly in diverse contexts the Parseval inequality which says the following.

Corollary 2.12 (Parseval)**.** Let $(e_n,\, n \geq 1)$ be a CONB. For any $x \in H$,

$$\|x\|_{H^2} = \sum_{n=1}^{\infty} \langle x,\, e_n \rangle_H^2 \text{ and } \langle x, y \rangle_H = \sum_{n=1}^{\infty} \langle x,\, e_n \rangle_H\, \langle y,\, e_n \rangle_H\, .$$

The classical example of Hilbert spaces is the space of square integrable functions from a measurable space $(E;\, \mu)$ into $\mathbf{R}$:

$$L^2(E; \mu) = \{f \,:\, E \to \mathbf{R},\, \int_E |f(x)|^2 \, \mathrm{d}\mu(x) < \infty\},$$

with the scalar product

$$\langle f,\, g \rangle_{L^2(E;\mu)} = \int_E f(x)\, g(x)\, \mathrm{d}\mu(x).$$

We already saw other interesting Hilbert spaces like $I_{1,2}$. In fact, this space is a particular instance of Hilbert spaces which appear very naturally in the theory of Gaussian processes.

Self-reproducing Hilbert spaces

Assume we are given a symmetric function R on $E \times E$ satisfying

$$\sum_{k,l=1}^{n} R(t_k,\, t_l)\, c_k\, c_l \geq 0$$

for any $n \geq 1$, any $t_1, \cdots, t_n \in E$ and any $c_1, \cdots, c_n \in \mathbf{R}$, with equality if and only if $c_k = 0$ for all k. Then, R is said to be symmetric positive definite kernel.

Definition 2.9. Consider $H_0 = \operatorname{span}(R(t, .),\, t \in E)$ and define an inner product on H_0 by

$$\langle R(t, .),\, R(s, .)\rangle_{H_0} = R(t, s). \tag{2.15}$$

Then, H is the completion of H_0 with respect to this inner product: The set of functions of the form

$$f(s) = \sum_{i=1}^{\infty} \alpha_i R(t_i, s)$$

for some denumerable family $(t_k,\, k \geq 1)$ of elements of E and some real numbers $(\alpha_k,\, k \geq 1)$ such that

$$\sum_{i=1}^{\infty} \alpha_i^2 R(t_i, t_i) < \infty.$$

Lemma 2.13 (Representation of an RKHS)**.** Assume that there exist a measure μ, a function $K : E \times E \to \mathbf{R}$ such that

$$R(s, t) = \int_E K(s, r)\, K(t, r)\, \mathrm{d}\mu(r), \tag{2.16}$$

and that the linear map defined by K is one-to-one:

$$\left(\forall t \in E, \int_E K(t, s) g(s)\, \mathrm{d}\mu(s) = 0\right) \Longrightarrow g = 0\ \mu - \text{a.s.}$$

Then the Hilbert space constructed above is equal to $K(L^2(E;\, \mu))$: The space of functions of the form

$$f(t) = \int_E K(t, s) g(s)\, \mathrm{d}\mu(s)$$

for some $g \in L^2(E;\, \mu)$, equipped with the inner product

$$\langle Kf,\, Kg\rangle_{K(L^2(E;\mu))} = \langle f, g\rangle_{L^2(E;\mu)}\,.$$

Proof. Since $K\Big(K(t,.)\Big)(s)=R(t,s)$,

$$K\Big(\sum_{k=1}^n \alpha_k K(t_k,.)\Big)=\sum_{k=1}^n \alpha_k R(t_k,.).$$

On the one hand,

$$\|\sum_{k=1}^n \alpha_k R(t_k,.)\|_H^2=\sum_{k=1}^n\sum_{l=1}^n \alpha_k\alpha_l R(t_k,t_l)$$

and on the other hand,

$$\begin{aligned}\|\sum_{k=1}^n \alpha_k K(t_k,.)\|^2_{L^2(E;\mu)}&=\int_E\Big(\sum_{k=1}^n \alpha_k K(t_k,s)\Big)^2\,\mathrm{d}\mu(s)\\&=\sum_{k=1}^n\sum_{l=1}^n \alpha_k\alpha_l\iint_{E\times E}K(t_k,s)K(t_l,s)\,\mathrm{d}\mu(s)\\&=\sum_{k=1}^n\sum_{l=1}^n \alpha_k\alpha_l R(t_k,t_l),\end{aligned}$$

in view of (2.16). □

Example 2.– Consider $E=[0,1]$, μ the Lebesgue measure and $R(t,s)=\min(t,s)$. It is easy to see that

$$\min(t,s)=\int_0^1 \mathbf{1}_{[0,t]}(r)\,\mathbf{1}_{[0,s]}(r)\,\mathrm{d}r.$$

This means that the RKHS defined by R is equal to $I_{1,2}$ since for $K(t,r)=\mathbf{1}_{[0,t]}(r)$,

$$Kf(t)=\int_0^1 K(t,r)f(r)\,\mathrm{d}r=I^1f(t).$$

Dual of an Hilbert space

In $\mathbf{R}^n$, it is known that any linear form on $\mathbf{R}^n$ is necessarily continuous and can be represented by a vector of $\mathbf{R}^n$: If T is a linear map from $\mathbf{R}^n$ to $\mathbf{R}$ then these exists $x_0\in\mathbf{R}^n$ such that

$$T(x)=\langle x,\,x_0\rangle_{\mathbf{R}^n}\,.$$

This result can be extended to Hilbert spaces.

Theorem 2.14. Let T be a continuous linear map from H into $\mathbf{R}$. There exists a unique $x_T\in H$ such that

$$T(x)=\langle x_T,\,x\rangle_H\,,\ \text{ for all } x\in H. \tag{2.17}$$

Moreover, the map

$$\begin{aligned}\iota_H\,:\,H^*&\longrightarrow H\\T&\longmapsto x_T\end{aligned}$$

is a (bijective) isometry.

Example 3.– According to Theorem 2.4, $I_{1,2} \subset \text{Hol}(1/2)$. Thus, the Dirac measure ε_a is a continuous linear map on $I_{1,2}$. Let x_a be its representation in $I_{1,2}$. We must have for any $f \in I_{1,2}$,

$$\varepsilon_a(f) = f(a) = f(a) - f(0) = \langle x_a, f \rangle_{I_{1,2}} = \int_0^1 \dot{x_a}(s)\dot{f}(s)\,\mathrm{d}s,$$

where $\dot{f} = (I^1)^{-1}f$ is the derivative of f. The sole candidate is $\dot{x}_a = \mathbf{1}_{[0,a]}$, hence $x_a(s) = a \wedge s$, i.e. $\iota_{I_{1,2}}(\varepsilon_a) = . \wedge a$.

Compact maps in Hilbert spaces

Definition 2.10. A linear map T between two Hilbert spaces H_1 and H_2 is said to be compact whenever the image of any bounded subset in H_1 is a relatively compact subset (i.e. its closure is compact) in H_2. It can be written: For any $h \in H_1$

$$Th = \sum_{n=1}^{\infty} \lambda_n \langle f_n, h \rangle_{H_1} g_n$$

where $(f_n, n \geq 1)$ and $(g_n, n \geq 1)$ are orthonormal sets of respectively H_1 and H_2. Moreover, $(\lambda_n, n \geq 1)$ is a sequence of positive real numbers with sole accumulation point zero. If for some rank N, $\lambda_n = 0$ for $n \geq N$, the operator is said to be of finite rank.

Among those operators, some will play a crucial rôle in the sequel.

Definition 2.11 (Trace class operators)**.** Let H be a Hilbert space and $(e_n, n \geq 1)$ be a CONB on H. A linear map A from H into itself is said to be trace-class whenever

$$\sum_{n\geq 1} |\langle Ae_n, e_n \rangle| < \infty.$$

Then, its trace is defined as

$$\text{trace}(A) = \sum_{n\geq 1} \langle Ae_n, e_n \rangle.$$

In the decomposition of Definition 2.10, this means that

$$\sum_{n=1}^{\infty} |\lambda_n| < \infty.$$

Definition 2.12 (Hilbert-Schmidt operators)**.** Let H_1 and H_2 be two Hilbert space and $(e_n,\, n \ge 1)$ (resp. $(f_p,\, p \ge 1)$) a CONB of H_1 (resp. H_2). A linear map A from H_1 into H_2 is said to be Hilbert-Schmidt whenever

$$\|A\|_{\text{HS}}^2 = \sum_{n\ge 1} \|Ae_n\|_{H_2}^2 = \sum_{n\ge 1}\sum_{p\ge 1} \langle Ae_n,\, f_p\rangle_{H_2}^2 < \infty.$$

If $H_1 = H_2$, in the decomposition of Definition 2.10, this means that

$$\sum_{n=1}^{\infty} \lambda_n^2 < \infty.$$

Note that a linear map from H into itself can be described by an infinite matrix: To characterize A, since H has a basis, it is sufficient to determine its values on this basis. This means that A is completly determined by the family $(\langle Ae_n,\, e_k\rangle_H\,,\, n,k \ge 1)$, which is nothing but a kind of an infinite matrix. We can also write

$$\langle Ae_n,\, e_k\rangle_H = \langle A,\, e_n \otimes e_k\rangle_{H\otimes H}\,,$$

so that A appears as a linear map on $H \otimes H$.

Theorem 2.15. If $H = L^2(\mu)$ and A is Hilbert-Schmidt then there exists a kernel which we still denote by $A : H \times H \to \mathbf{R}$ such that for any $f \in H$,

$$Af(x) = \int_H A(x,y)\, f(y)\, \mathrm{d}\mu(y)$$

and

$$\|A\|_{\text{HS}}^2 = \iint_{H\times H} |A(x,y)|^2\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y).$$

Theorem 2.16 (Composition of Hilbert-Schmidt maps)**.** With the same notations as above, the composition of two Hilbert-Schmidt is trace class.
Actually, this is an equivalence: A trace-class map can always be written as the composition of two Hilbert-Schmidt operators. Moreover,

$$|\operatorname{trace}(A\circ B)| \le \sum_{n\ge 1} \left|\langle A\circ Be_n,\, e_n\rangle_H\right| \le \|A\|_{\text{HS}}\|B\|_{\text{HS}}. \tag{2.18}$$

Lemma 2.17 (Composition of integral maps)**.** If $H = L^2(\mu)$ and A, B are Hilbert-Schmidt maps on H. Then, $B \circ A$ is trace-class and

$$\operatorname{trace}(B\circ A) = \iint_{H\times H} B(x,y)A(y,x)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y).$$

Proof. We must verify the finiteness of

$$\sum_{n\geq 1} |\left\langle BAe_n,\, e_n\right\rangle_H|.$$

By the definition of the adjoint, applying twice the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{n\geq 1} |\left\langle BAe_n,\, e_n\right\rangle_H| = \sum_{n\geq 1} |\left\langle Ae_n,\, B^*e_n\right\rangle_H| \leq \sum_{n\geq 1} \|Ae_n\|_H \|B^*e_n\|_H \\ \leq \left(\sum_{n\geq 1} \|Ae_n\|_H^2\right)^{1/2} \left(\sum_{n\geq 1} \|B^*e_n\|_H^2\right)^{1/2} = \|A\|_{\mathrm{HS}}\|B^*\|_{\mathrm{HS}}.\end{aligned}$$

The Parseval identity (twice) yields

$$\begin{aligned}\operatorname{trace}(B\circ A) &= \sum_{n\geq 1} \left\langle Ae_n,\, B^*e_n\right\rangle_H = \sum_{n\geq 1}\sum_{k\geq 1} \left\langle Ae_n,\, e_k\right\rangle_H \left\langle B^*e_n,\, e_k\right\rangle_H \\ &= \sum_{n\geq 1}\sum_{k\geq 1} \left\langle A, e_k\otimes e_n\right\rangle_{H\otimes H} \left\langle B^*, e_k\otimes e_n\right\rangle_{H\otimes H} = \left\langle A,\, B^*\right\rangle_{H\otimes H}.\end{aligned}$$

By the identification of A, B and their kernel,

$$\begin{aligned}\left\langle A,\, B^*\right\rangle_{H\otimes H} &= \iint_{H\times H} A(x,y)B^*(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &= \iint_{H\times H} A(x,y)B(y,x)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).\end{aligned}$$

The proof is thus complete. □

Example 4 (Hilbert-Schmidt embeddings in fractional Liouville spaces).– Since $I^\alpha\circ I^\alpha = I^{\alpha+\alpha}$, we have

$$I_{\alpha,2}\subset I_{\alpha,2} \text{ for any } \alpha>\alpha.$$

Lemma 2.18. The embedding emb of $I_{\alpha,2}$ into $I_{\alpha,2}$ is Hilbert-Schmidt if and only if $\alpha-\alpha>1/2$.

Proof. Let $(e_n,\, n\geq 1)$ be CONB of $L^2([0,1])$ and set $h_n = I^\alpha e_n$. Then $(h_n,\, n\geq 1)$ is a CONB of $I_{\alpha,2}$. We must prove that

$$\sum_{n=1}^{\infty} \|\operatorname{emb}(h_n)\|_{I_{\alpha,2}}^2 < \infty.$$

By the very definition of the norm in $I_{\alpha,2}$, this is equivalent to show

$$\sum_{n=1}^{\infty} \|I^{\alpha-\alpha}(e_n)\|_{L^2}^2 < \infty.$$

But this latter sum turns to be equal to the Hilbert-Schmidt norm of $I^{\alpha-\alpha}$ viewed as a linear map from L^2 into itself. In view of Proposition 2.15, $I^{\alpha-\alpha}$ is Hilbert-Schmidt if and only if

$$\iint_{[0,1]^2} |t-s|^{2((\alpha-\alpha)-1)}\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

This only happens if $\alpha-\alpha>1/2$. □

§ 2 Exercises

Exercice 2.1.–
From Theorem 2.4, we know that $I_{\alpha,2} \subseteq L^2$ for any $\alpha > 0$.

Show that this embedding is Hilbert-Schmidt if and only if $\alpha > 1/2$.

Exercice 2.2.–
For $\alpha > 1/2$, $I_{\alpha,2} \subseteq \mathrm{Hol}(\alpha - 1/2) \subset \mathcal{C}$ hence, the Dirac measure ϵ_τ belongs to $I^*_{\alpha,2}$. Let j_α be the canonical isometry between $I^*_{\alpha,2}$ and $I_{\alpha,2}$.

Show that

$$j_\alpha(\epsilon_\tau) = \frac{1}{\Gamma(\alpha)} I^\alpha\Big((\tau - .)^{\alpha-1} \Big).$$

Exercice 2.3.–
Following the proof of Theorem 2.6, show that $(S_n,\, n \ge 0)$ as defined in (2.4) is convergent in $I_{\alpha,2}$ for $\alpha < 1/2$.

It is important to remark that $(\dot{h}_n,\, n \ge 0)$ is an orthonormal family of L^2.

Show that for any $z \in I_{\alpha,2}$,

$$\mathbf{E}\left[e^{i\langle z, \sum_n X_n h_n \rangle_{I_{\alpha,2}}} \right] = \exp(-\frac{1}{2} \langle V_\alpha z, z \rangle_{I_{\alpha,2}})$$

where

$$V_\alpha = I^\alpha \circ I^{1-\alpha} \circ (I^{1-\alpha})^* \circ (I^\alpha)^{-1}.$$