Probabilistic Databases: Width-Based Approaches

Antoine Amarilli

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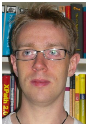

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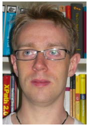
*Conversely, there is a query Q for which $\mathrm{PQE}(Q)$ is intractable on **any** input instance family of unbounded treewidth (under some technical assumptions)*



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Computational complexity as a function of w
(the query Q is **fixed**)

Monadic second-order logic (MSO)

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- $x \to y$ means "x is the predecessor of y"

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- **Monadic second-order logic (MSO):** adds **quantifiers over sets**
  - $\exists S\ \forall x\ S(x)$ means *"there is a set S containing every element x"*
  - Can express **transitive closure** $x \to^* y$, i.e., *"x is before y"*
  - $\forall x\, P_{\color{pink}\bullet}(x) \Rightarrow \exists y\, P_{\color{blue}\bullet}(y) \wedge x \to^* y$
    means *"There is a blue node after every pink node"*

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Translate the query Q to a deterministic word automaton

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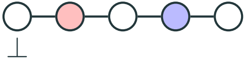
w: 

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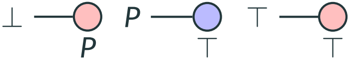
- Transitions (examples): 

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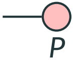
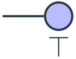
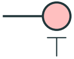
- **Transitions** (examples): $\bot$  P  $\top$ 

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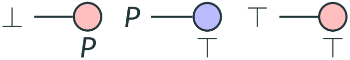
- **Transitions** (examples): 

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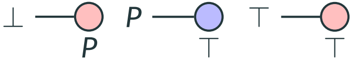
- **Transitions** (examples): 

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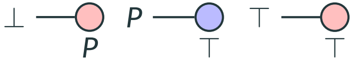
- **Transitions** (examples): 

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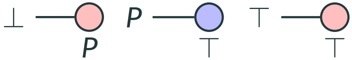
- **Transitions** (examples): 

Theorem (Büchi, 1960)

*MSO and **word automata** have the same **expressive power** on words*

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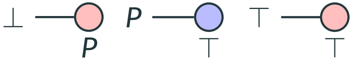
- Transitions (examples): 

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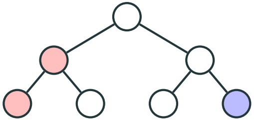
Corollary

Query evaluation of MSO on words is in linear time (in data complexity)

Non-probabilistic query evaluation on trees

Database: a **tree** T where nodes have a color from an alphabet



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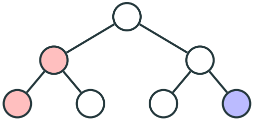

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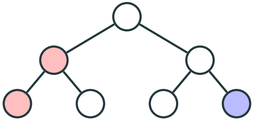

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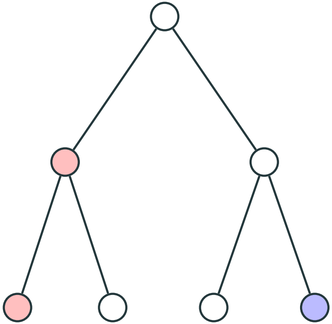

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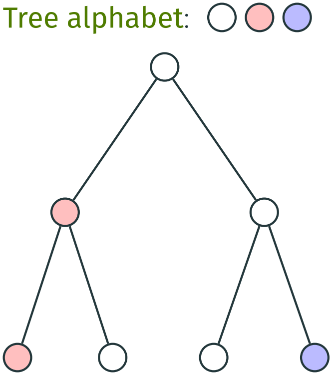

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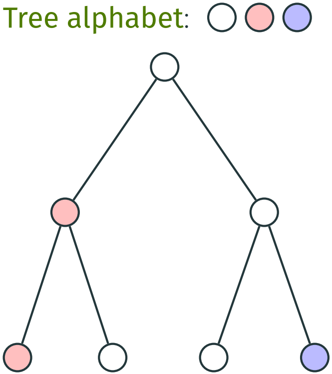

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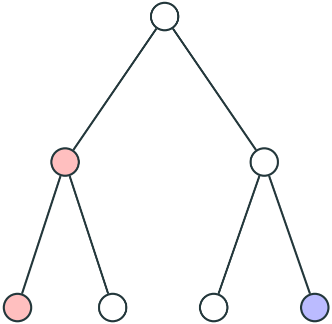

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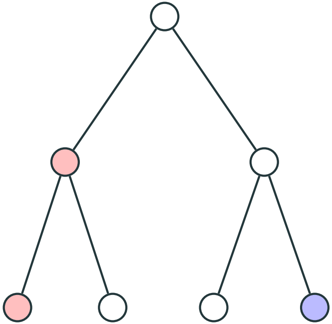

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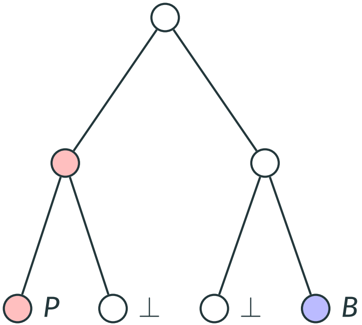

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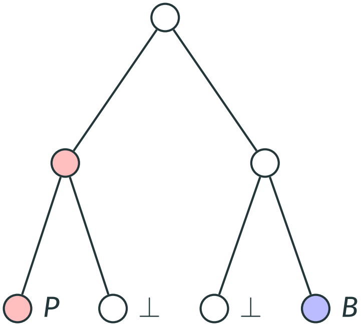

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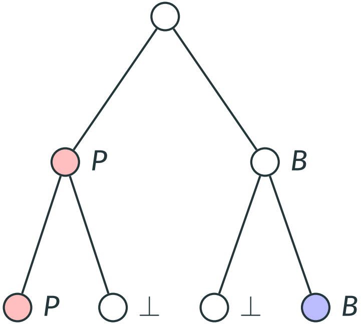

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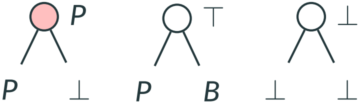

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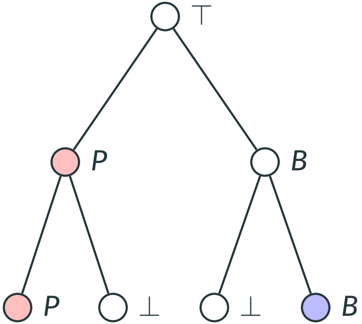

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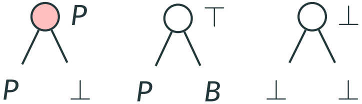

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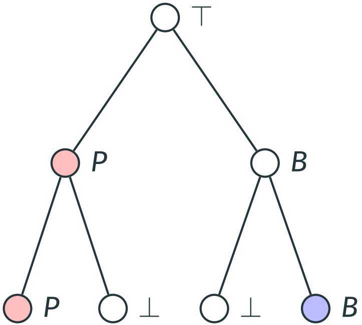

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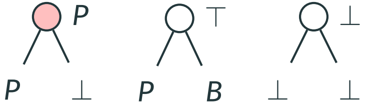

Theorem ([Thatcher and Wright, 1968])

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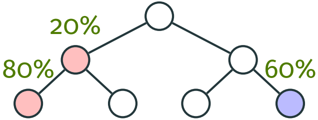

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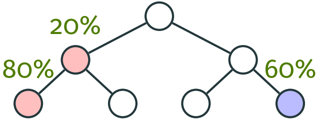

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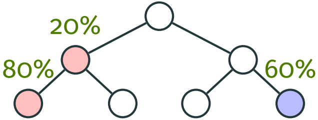

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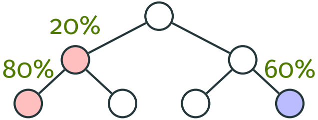

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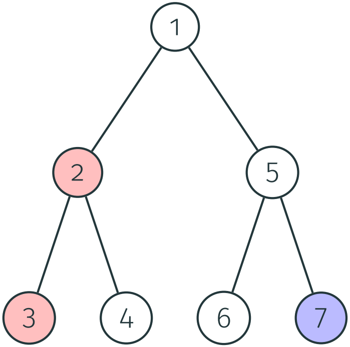
Theorem

*For any fixed **MSO query** Q, the problem $\mathrm{PQE}(Q)$ on trees is in **linear time** assuming constant-time arithmetics*

Uncertain trees: capturing how the query result depends on the choices



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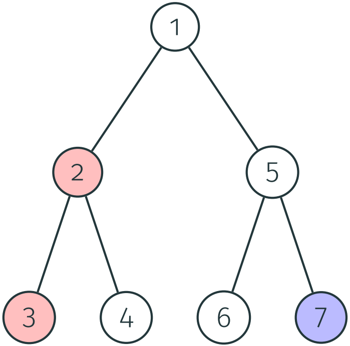

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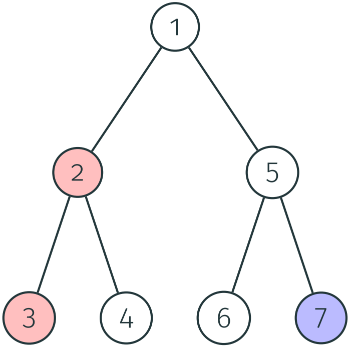

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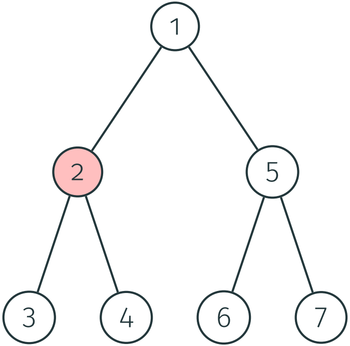

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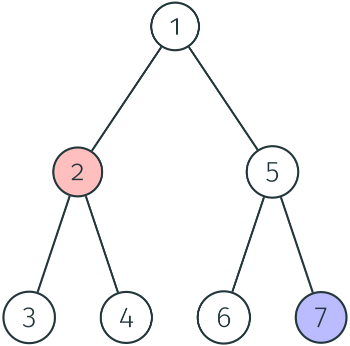

A **valuation** of a tree decides whether to **keep** (1) or **discard** (0) node labels

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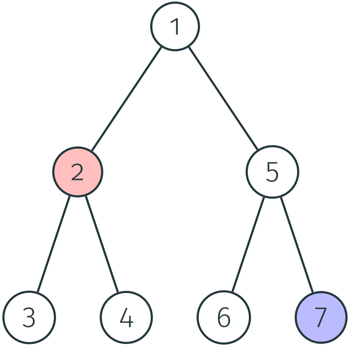

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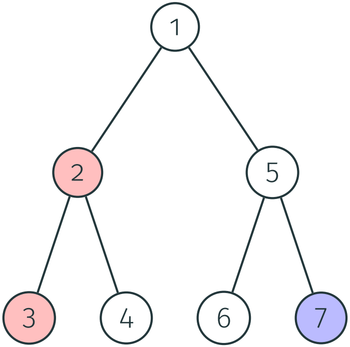

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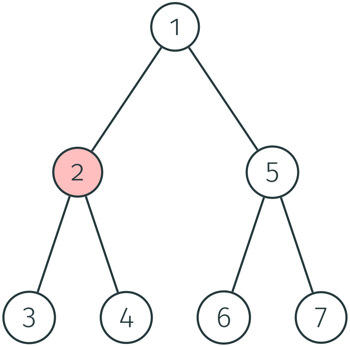

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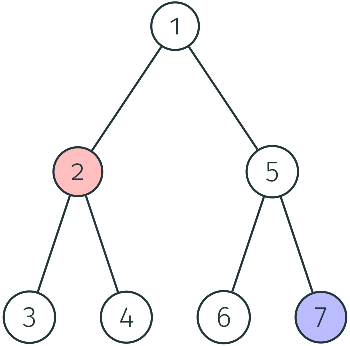

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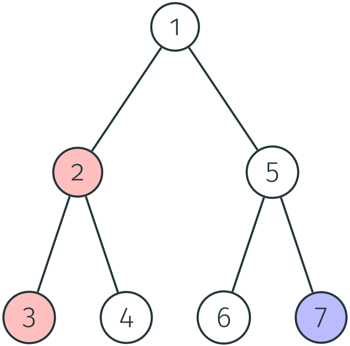
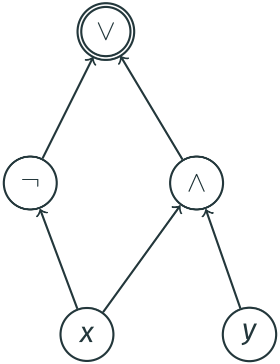


A **valuation** of a tree decides whether to **keep** (1) or **discard** (0) node labels

Q: *"Is there both a pink and a blue node?"*

$\rightarrow$ This is a so-called **Boolean provenance circuit** on the "color facts" of the tree nodes!

Boolean circuit



- Directed acyclic graph of gates

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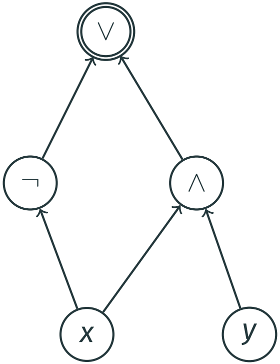

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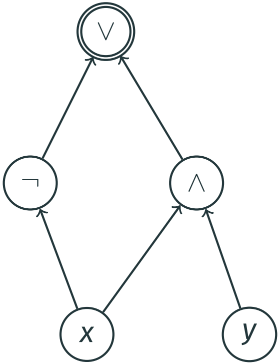

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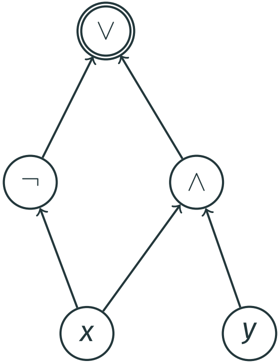

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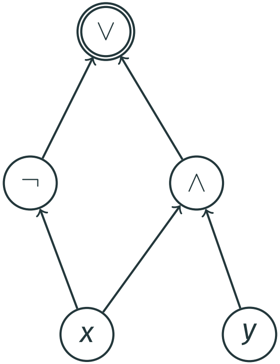

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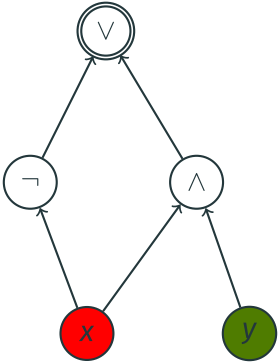

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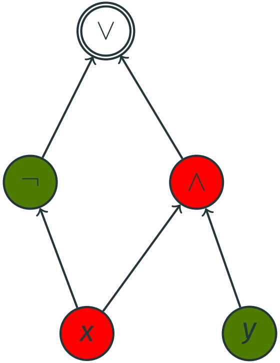

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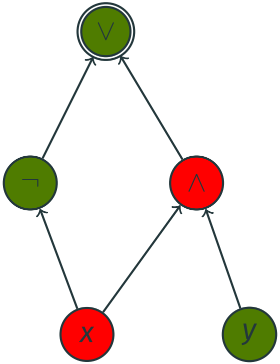

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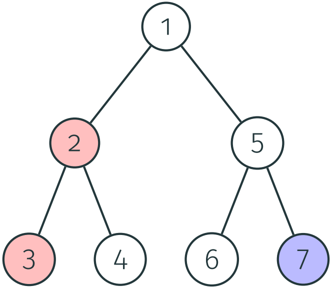
Example: Provenance circuit



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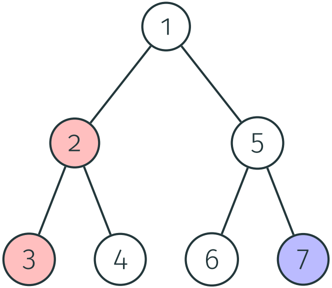

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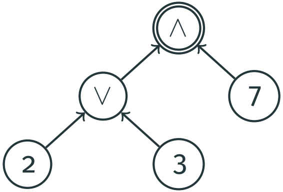

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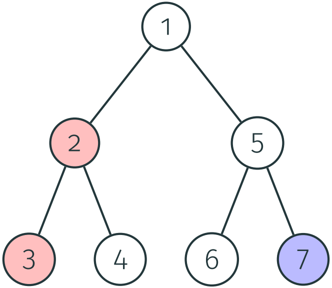

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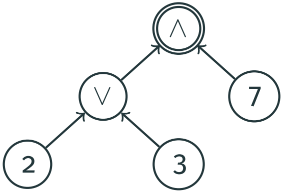

Formal definition of provenance circuits:

- Boolean query Q, uncertain tree T, circuit C
- **Variable gates** of C: nodes of T
- **Condition:** Let ν be a valuation of T, then $\nu(C)$ iff $\nu(T)$ satisfies Q

Building provenance circuits on trees

Theorem

*For any bottom-up **tree automaton** A and input **tree** T,*
*we can build a **provenance circuit** of A on T in $O(|A| \times |T|)$*

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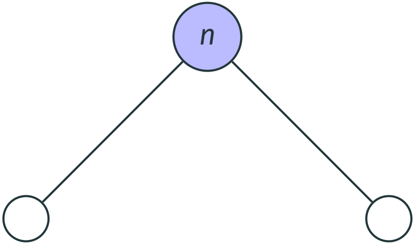

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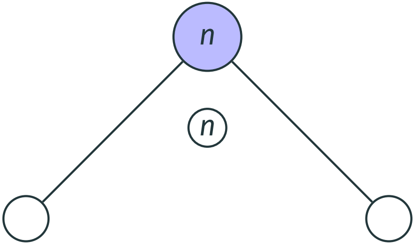

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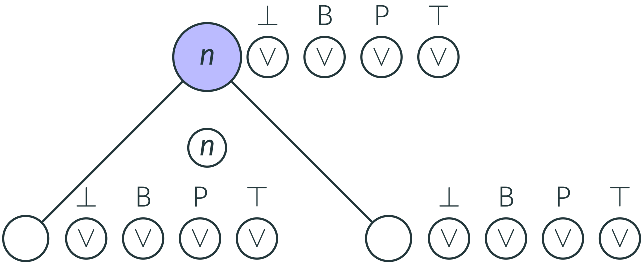

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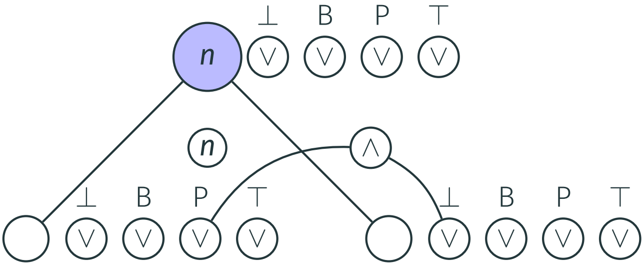

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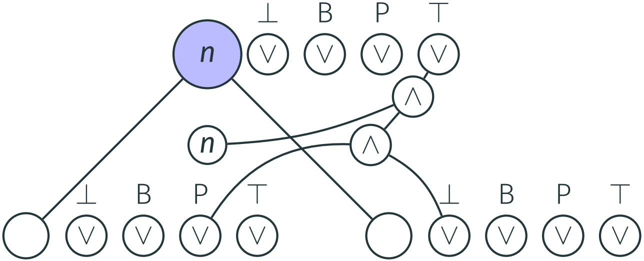

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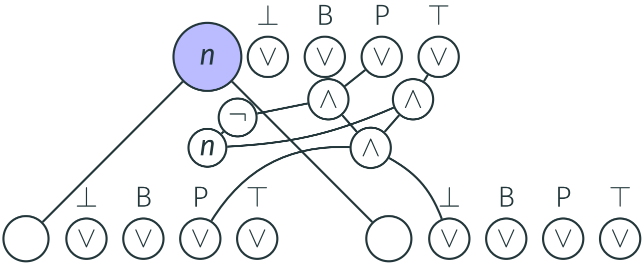

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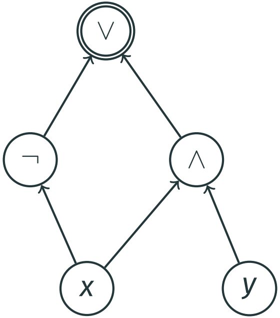

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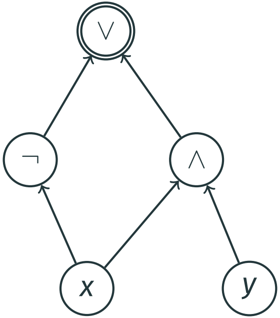

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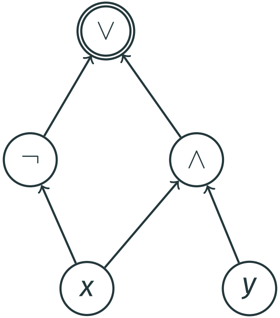

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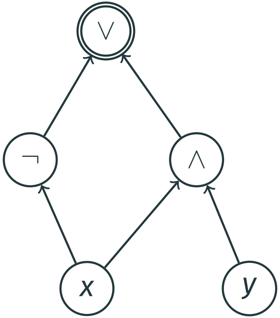

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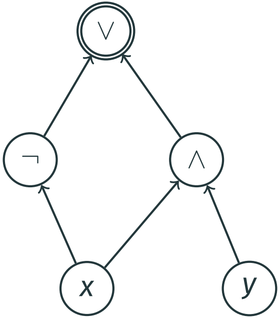

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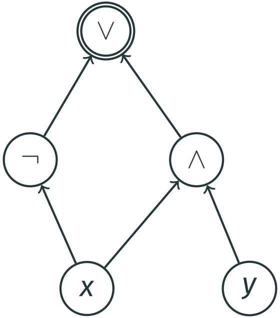

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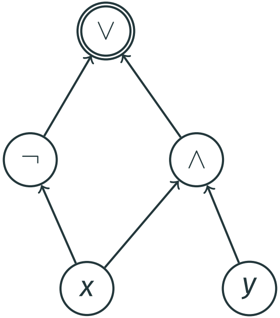

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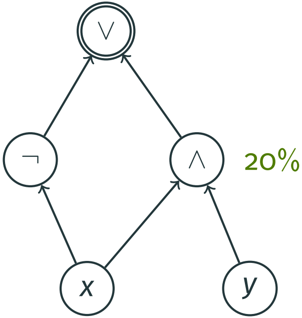

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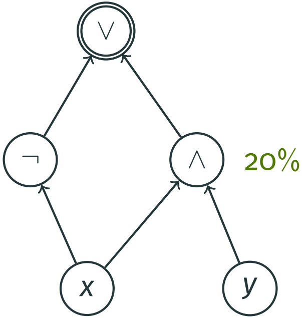

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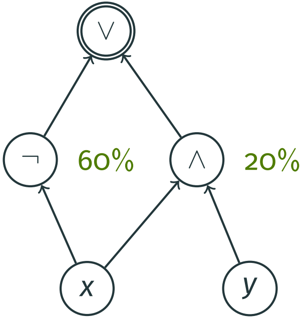

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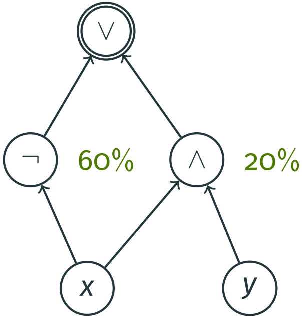

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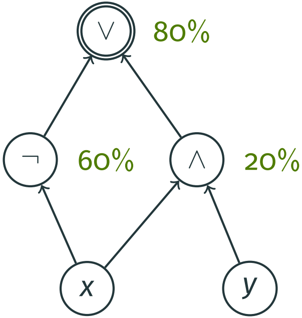

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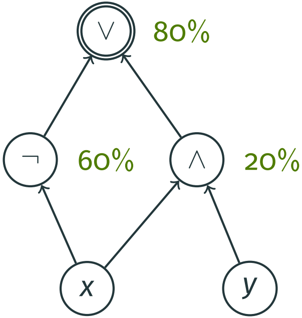

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- Let's focus on a **restricted class** of circuits that satisfies these conditions

d-DNNFs

The circuit is a d-DNNF...

d-DNNFs

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- $\neg$ gates only have variables as inputs

d-DNNFs

The circuit is a **d-DNNF**...

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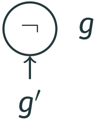

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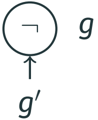

$$P(g) := 1 - P(g')$$

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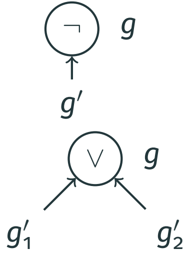
- $\wedge$ gates are all on **independent** inputs

... so probability computation is **easy**!



$$P(g) := 1 - P(g')$$

d-DNNFs

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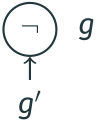

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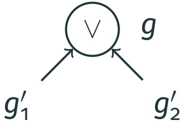

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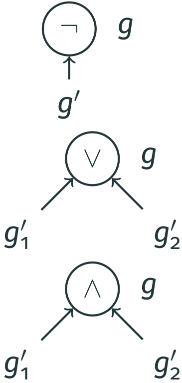

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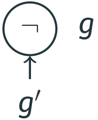

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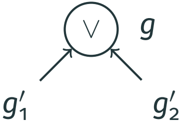
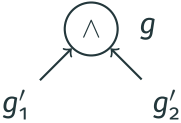


$$P(g) := P(g_1') + P(g_2')$$



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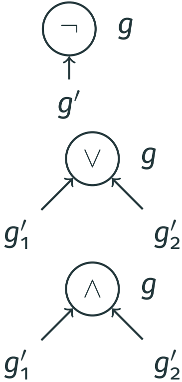

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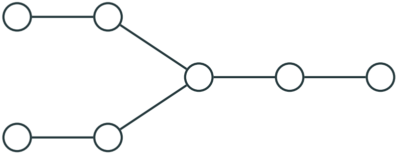
Lemma

The provenance circuit computed in our construction is a d-DNNF

Treewidth

We have shown tractability of PQE on **trees**; let us extend to **bounded treewidth**

Treewidth by example:



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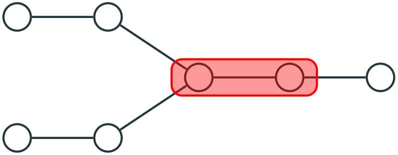

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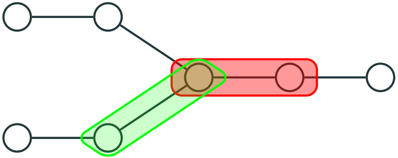

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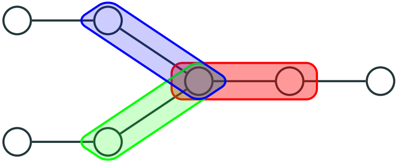

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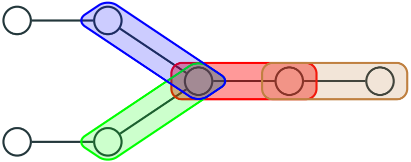

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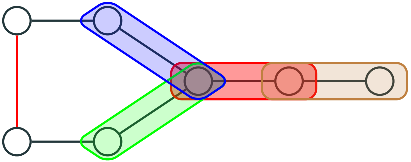

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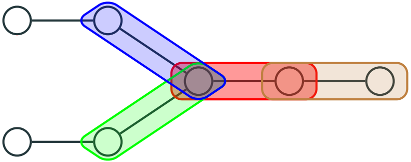

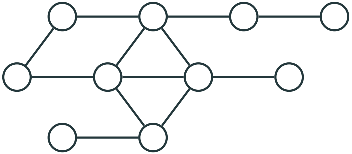

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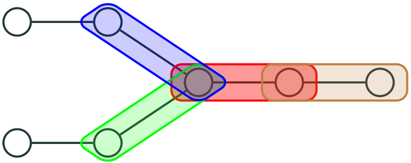

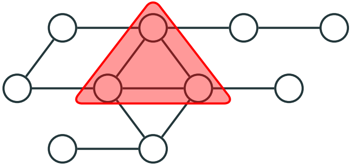

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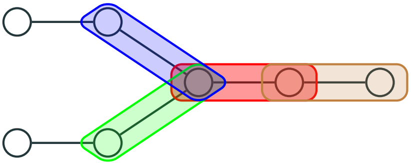

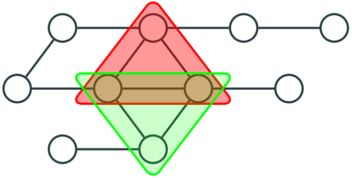

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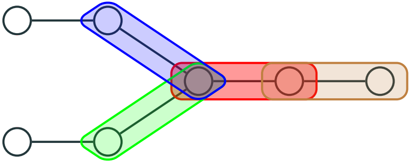

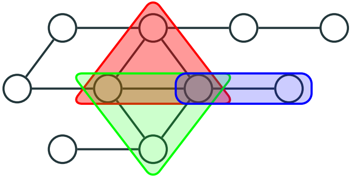

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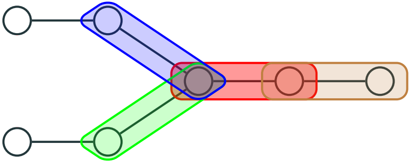

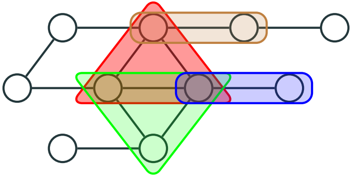

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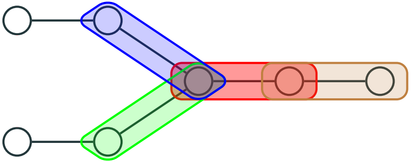

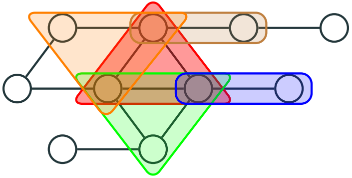

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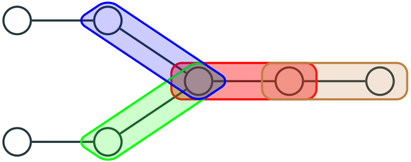

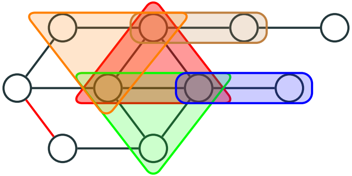

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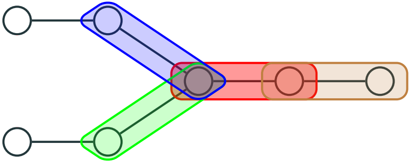

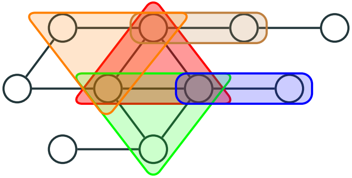

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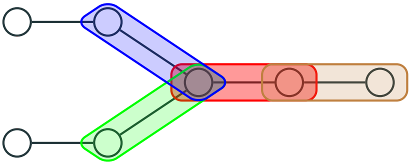

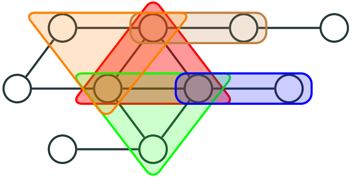

- **Trees** have treewidth 1
- **Cycles** have treewidth 2
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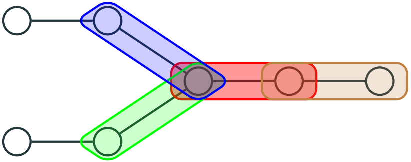

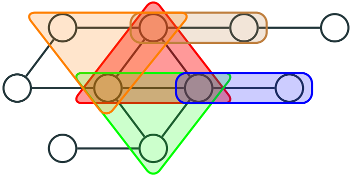

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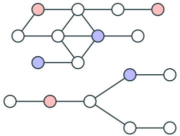
$\rightarrow$ **Treelike**: the **treewidth** is bounded by a **constant**

Courcelle's theorem and extension to PQE

Treelike **data**



MSO query

$$\exists x\, y$$
$$P_{\bigcirc}(x) \wedge P_{\bigcirc}(y)$$

Courcelle's theorem and extension to PQE

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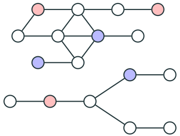

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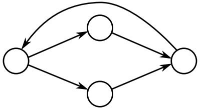
Tree **automaton**



Courcelle's theorem and extension to PQE

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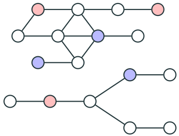
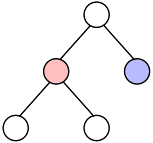


linear →

Tree **encoding**



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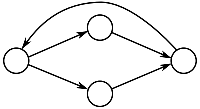
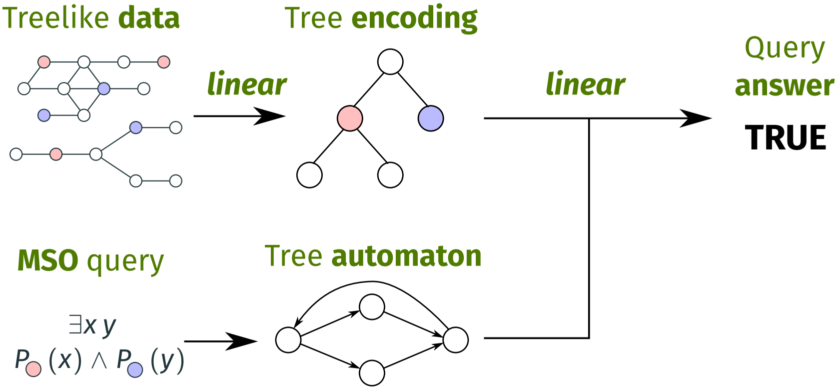


Courcelle's theorem and extension to PQE



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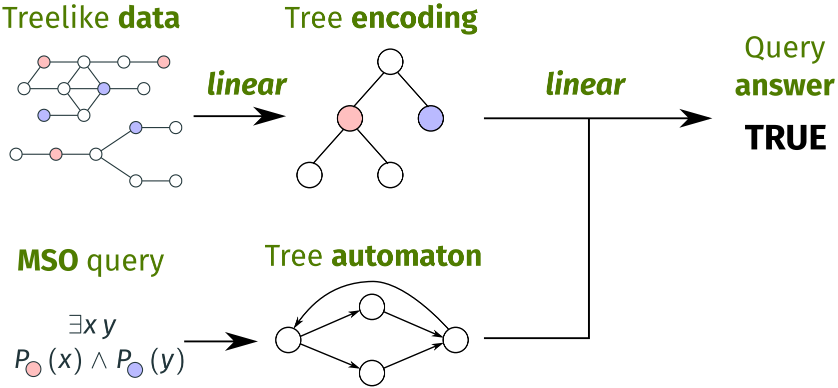
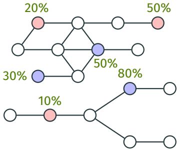


Theorem ([Courcelle, 1990])

For any fixed Boolean MSO query Q *and* $k \in \mathbb{N}$*, given a database* D *of treewidth* $\leq k$*, we can compute in linear time in* D *whether* D *satisfies* Q

Courcelle's theorem and extension to PQE

Probabilistic treelike **data**



MSO query

$$\exists x\, y \quad P_{\text{red}}(x) \land P_{\text{blue}}(y)$$

Courcelle's theorem and extension to PQE

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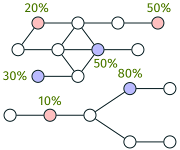

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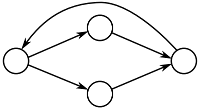
$\longrightarrow$

Tree **automaton**



Courcelle's theorem and extension to PQE

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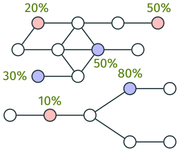

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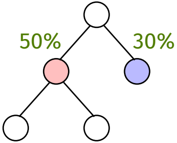

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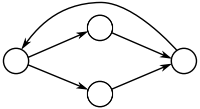
Tree **automaton**



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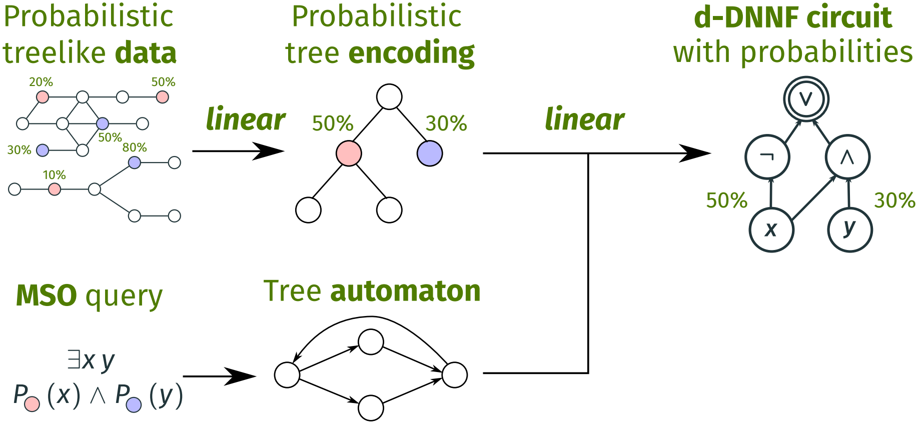

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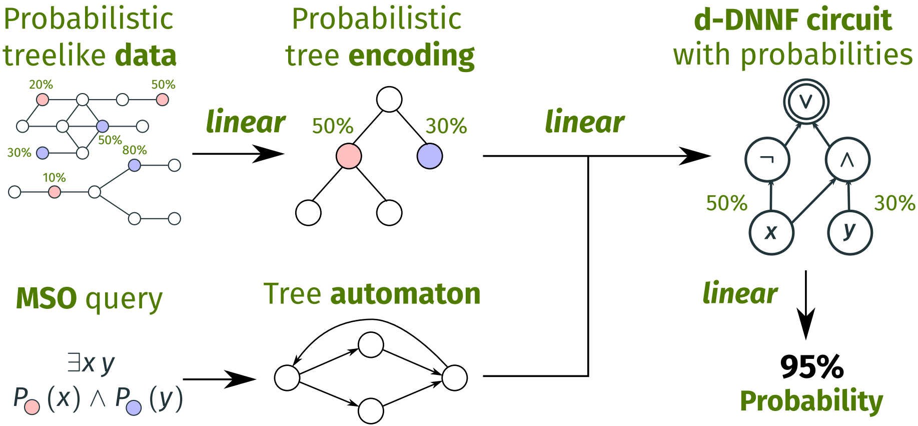

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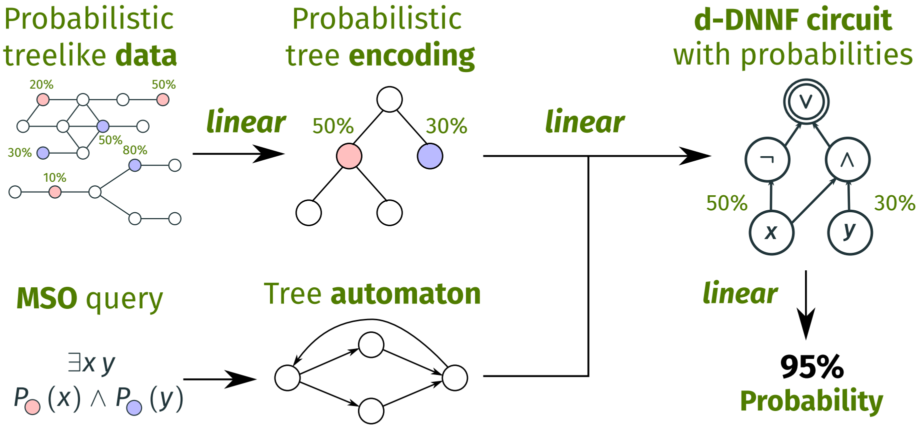

Theorem (A., Bourhis, Senellart, 2015, 2016)

For any fixed Boolean MSO query $\mathbf{Q}$ *and* $\mathbf{k} \in \mathbb{N}$, *given a database* $\mathbf{D}$ *of treewidth* $\leq \mathbf{k}$, *we can solve the PQE problem in linear time (assuming constant-time arithmetics)*

Why is this a dichotomy? Where’s the lower bound?

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Theorem (A., Bourhis, Senellart, 2016)

For any arity-two signature, there is a first-order query Q *such that for any constructible unbounded-treewidth family* $\mathcal{I}$ *of probabilistic graphs, the PQE problem for* Q *and* $\mathcal{I}$ *is #P-hard under RP reductions*

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$\rightarrow$ This result does **not** generalize to higher-arity!

Why is this a dichotomy? Where's the lower bound?

Theorem (A., Bourhis, Senellart, 2016)

For any arity-two signature, there is a ***first-order*** *query* $\mathbf{Q}$ *such that for any constructible* ***unbounded-treewidth*** *family* $\mathcal{I}$ *of probabilistic graphs, the PQE problem for* $\mathbf{Q}$ *and* $\mathcal{I}$ *is* ***#P-hard*** *under RP reductions*

- **Family:** an infinite set of graphs allowed as input (with arbitrary probabilities) so in particular **closed under subgraphs**
- **Unbounded-treewidth**: for all $k \in \mathbb{N}$, there is $I_k \in \mathcal{I}$ of treewidth $\geq k$
- **Constructible:** given k, we can **compute** such an instance I_k in PTIME
- **Under RP reductions:** reduce in PTIME with high probability

$\rightarrow$ This result does **not** generalize to higher-arity!

$\rightarrow$ Proof idea: **extract wall graphs as topological minors** ([Chekuri and Chuzhoy, 2014]) and use them for a lower bound

References i

Amarilli, A., Bourhis, P., and Senellart, P. (2015).
Provenance circuits for trees and treelike instances.
In *ICALP*.

Amarilli, A., Bourhis, P., and Senellart, P. (2016).
Tractable lineages on treelike instances: Limits and extensions.
In *PODS*.

Chekuri, C. and Chuzhoy, J. (2014).
Polynomial bounds for the grid-minor theorem.
In *STOC*.

References ii

Courcelle, B. (1990).
The monadic second-order logic of graphs. I. Recognizable sets of finite graphs.
Inf. Comput., 85(1).

Thatcher, J. W. and Wright, J. B. (1968).
Generalized finite automata theory with an application to a decision problem of second-order logic.
Mathematical systems theory, 2(1).