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WHEN DOES THE RAMER FORMULA LOOK LIKE THE GIRSANOV FORMULA?

By M. Zakai[1] and O. Zeitouni[2]

Technion–Israel Institute of Technology, Haifa, Israel

Let $\{B, H, P_0\}$ be an abstract Wiener space and for every real ρ, let $T_\rho \omega = \omega + \rho F(\omega)$ be a transformation from B to B. It is well known that under certain assumptions the measures induced by T_ρ or T_ρ^{-1} are mutually absolutely continuous with respect to P_0 and the density function is represented by the Ramer formula. In this formula, the Carleman–Fredholm determinant $\det_2(I_H + \rho\nabla F)$ appears as a factor. We characterize the class of ∇F for which a.s.-P_0, $\det_2(I_H + \rho\nabla F) = 1$ for all ρ in an open subset of $\mathbb{R}$, in which case the form of Ramer's expression reduces to the familiar Cameron–Martin–Maruyama–Girsanov form. The proof is based on a characterization of quasinilpotent Hilbert–Schmidt operators.

1. Introduction. Let $\{B, H, P_0\}$ be an abstract Wiener space, let ρ be a real parameter and $T_\rho \omega = \omega + \rho F(\omega)$ be a transformation from B to B. By the work of Ramer [4] and Kusuoka [2], it is well known that if: (i) $I + \rho F(\omega)$ is bijective as an operator from B to B and $F(\omega)$ transforms B to H; (ii) $F(\omega)$ possesses a weak-H derivative ∇F which is a.s. Hilbert–Schmidt from H to H, and is sufficiently smooth (cf. Theorem 6.4 of [2]); and (iii) $(I_H + \rho\nabla F(\omega))$ is a.s. an invertible operator from H to itself, then both T_ρ and T_ρ^{-1} induce absolutely continuous transformations of measure and, setting

$$Y(\omega) = \left(d\left(T_\rho^{-1}\right)^* P_0/dP_0\right)(\omega),$$

where

$$\left(T_\rho^{-1}\right)^* P_0(A) = P_0\left\{\omega: T_\rho^{-1}\omega \in A\right\},$$

then

$$Y(\omega) = \left|\det_2\left(I_H + \rho\nabla F(\omega)\right)\right| \cdot \exp\left\{-\delta F(\omega) - \tfrac{1}{2}\langle F, F\rangle_H\right\}, \tag{1}$$

where δF denotes the divergence of F and $\det_2$ denotes the Carleman–Fredholm determinant. Note that in the classical Wiener case, if $\nabla_\theta F$ is adapted to the subsigma fields induced by $\{W_\eta, \eta \le \theta\}$ for all $\theta \in [0,1]$, then the divergence δ reduces to the Ito integral (of $d\nabla_\theta F/d\theta$) and $\det_2(I_H + \rho\nabla F) = 1$ for all ρ (cf., e.g., [3]). Therefore if for some ρ, $\det_2(I_H + \rho\nabla F) = 1$ a.s. $-$ P_0, then it is natural to say that for this ρ the Ramer formula (1) looks like the Girsanov formula (we remark that the Girsanov formula

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should probably be referred to as the Cameron–Martin–Maruyama–Girsanov formula; however for clarity we will stick to Girsanov formula). The problem of the characterization of F or ∇F for which, ρ given and fixed, the Ramer formula looks like the Girsanov formula does not seen to be easy. We therefore modify the problem by asking for the characterization of the ∇F for which the Ramer formula looks like the Girsanov formula for all $\rho \in \mathscr{O}$, where $\mathscr{O}$ is an open subset of the real line. The purpose of this note is to present such a characterization (cf. the corollary in Section 3).

2. A preliminary result. Let $\mathscr{H}$ be a Hilbert space and A a Hilbert–Schmidt (HS) operator from $\mathscr{H}$ to $\mathscr{H}$. Recall that such an operator is said to be quasinilpotent if $\lim_{n\to\infty} |A^n|^{1/n} = 0$ and this is equivalent to $\sigma(A) = 0$, where $\sigma(A)$ denotes the spectrum of A (cf. Lemma VII.3.4 of [1]). Let A, B be HS and let μ_i denote the eigenvalues of $A \cdot B$ repeated according to their multiplicity. Then the sum $\sum \mu_i$ converges absolutely and Trace$(A, B) = \sum \mu_i$ (cf. XI.6 of [1]). Theorem XI.6.24 of [1] states that if A is quasinilpotent, then Trace$(A, A) = 0$. The converse to this theorem is obviously not true (e.g., take A to be a 3×3 real matrix with eigenvalues $\sqrt{2}, i, -i$). We show that under additional conditions, the reverse direction does, however, hold true as follows:

THEOREM. *Let A be a HS operator from $\mathscr{H}$ to $\mathscr{H}$, denote by $\lambda_{\max}$ the maximal modulus of its eigenvalues and let $C > \lambda_{\max}$. Then the following are equivalent*:

(a) *A is quasinilpotent* [*i.e.*, $\sigma(A) = 0$ *or* $|A^n|^{1/n} \to_{n\to\infty} 0$].

(b) Trace$(A^k, A^k) = 0$ *for all* $k \geq 1$.

(c) Trace$(g(A), f(A)) = 0$ *for all functions* $g(z)$, $f(z)$ *which are analytic on* $|z| \leq C$ *and vanish at* $z = 0$.

(d) $\det_2(I + \rho A) = 1$ *for all* $\rho \in \mathbb{R}$.

(e) $\det_2(I + \rho A) = 1$ *for all* ρ *in an open set* $\mathscr{O}$.

(f) $(I + \rho A)^{-1} = I + B_\rho$ *for all* ρ *in some open set* $\mathscr{O}$ *and* B_ρ *is HS and quasinilpotent for all* ρ *in* $\mathscr{O}$.

PROOF. We remark that most of the proof follows results in the literature. The part which is novel is the proof of the implication (b) $\Rightarrow$ (a).

Note that since A is HS, all its eigenvalues λ_i are countable, of finite multiplicity and $\lambda_i \to 0$ as $i \to \infty$ and $\sigma(A) = \{\lambda_i, i = 1, 2, \dots\}$. Moreover, if $(I + \rho A)$ is invertible, namely, $\rho \notin \sigma(A)$, setting

$$(2) \qquad (I + \rho A)^{-1} = I + B_\rho$$

and using $(I + \rho A)(I + B_\rho) = I$ yields

$$I + B_\rho = I - \rho A \cdot (I + \rho A)^{-1},$$

therefore [since $(I + \rho A)^{-1}$ is bounded] B_ρ is also HS.

(a) ⇒ (f): We have to show that B_ρ is quasinilpotent. Assuming otherwise, let λ be a nonzero eigenvalue of B_ρ and v the corresponding eigenvector; $\lambda \neq -1$ since $I + B_\rho$ is invertible. Then, substituting in (2) yields that

$$-\frac{\lambda}{(1+\lambda)\rho}$$

is a nonzero eigenvalue of A (v being the corresponding eigenvector), which is impossible since $\sigma(A) = 0$.

(f) ⇒ (a): Follows by the same arguments as (a) ⇒ (f).

(a) ⇒ (c): Follows directly from Theorem XI.6.25 of [1].

(c) ⇒ (b): Obvious.

(a) ⇒ (d): Follows directly from part a of theorem 9.2 of [5].

(d) ⇒ (e): Obvious.

(e) ⇒ (b): By Theorem XI.6.26 of [1], $\det_2(I + \rho A)$ is an entire function of ρ. Hence, since $\det_2(I + \rho A) = 1$ on an interval in the complex domain it is equal to 1 for all ρ in the complex domain. On the other hand, $\det_2(I + \rho A)$ possesses the following series expansion (cf. page 108 of [5]):

$$\det_2[I + \rho A] = \exp \sum_2^\infty \frac{(-1)^{m+1}}{m} \rho^m \operatorname{Trace}(A, A^{m-1})$$

and (b) follows by the analyticity of $\det_2 [I + \rho A]$.

(b) ⇒ (a): Let (c′) denote the condition

(c′) Trace $(g(A^2), f(A^2)) = 0$ for all functions $g(z)$, $f(z)$ which are analytic on $|z| \leq C$ and vanish at $z = 0$.

Note first that (b) ⇒ (c′). Indeed, (b) implies that Trace$(A^{2k}, A^{2m}) = 0$ for all $m, k \geq 1$. The continuity of the Trace operator and the analyticity of f and g yield (c′). We show next that (c′) ⇒ (a): Set $f(z) = z$ and

$$g_\varepsilon(z) = \frac{z - \lambda_1^2 + \varepsilon}{z - \lambda_1^2 - \varepsilon} \cdot z,$$

where λ_1 is one of the eigenvalues of A with modulus $\lambda_{\max}$. Note that $g_\varepsilon(\lambda_1^2) = -\lambda_1^2$ and

$$g_\varepsilon(z) - z = \frac{2z\varepsilon}{z - \lambda_1^2 - \varepsilon}$$

and therefore $g_\varepsilon(z) \to_{\varepsilon \to 0} z$ for all $z \neq \lambda_1^2$, uniformly in z outside a small disk around λ_1^2. Hence, by XI.6.25 of [1],

$$0 = \operatorname{Trace}(A^2, A^2) = \sum_{i=1}^\infty \lambda_i^4,$$

and on the other hand, denoting by m the multiplicity of $\pm\lambda_1$,

$$\operatorname{Trace}(A^2, g_\varepsilon(A^2))$$

$$= -m\lambda_1^4 + \sum_{m+1}^\infty \lambda_i^4 + 2\varepsilon \sum_{m+1}^\infty \frac{\lambda_i^3}{\lambda_i^2 - \lambda_1^2 - \varepsilon} \to_{\varepsilon \to 0} -m\lambda_1^4 + \sum_{m+1}^\infty \lambda_i^4.$$

Consequently, $\lambda_1^4 = 0$ which proves that (b) ⇒ (a) and completes the proof of the theorem. □

3. Ramer's formula. The theorem of the previous section yields the following characterization:

COROLLARY. *Let $T_\rho\omega = \omega + \rho F(\omega)$, and assume that for all $\rho \in \mathscr{O}$, where $\mathscr{O}$ is an open set in $\mathbb{R}$, T_ρ satisfies* (i)–(iii) *from before. Then a necessary and sufficient condition that for all $\rho \in \mathscr{O}$,*

$$Y(\omega) = \exp\left\{-\rho\delta F - \frac{\rho^2}{2}\langle F, F\rangle_H\right\}$$

[namely, $\det_2(I_H + \rho\nabla F) = 1$] is that any of the conditions of the theorem hold for all ρ in $\mathscr{O}$ for almost all (P_0) ω. Furthermore, since $(T_\rho^{-1})^*P_0 \sim P_0$ and T_ρ bijective imply that $(T_\rho)^*P_0 \sim P_0$ with $X(\omega) = Y^{-1}(T_\rho\omega)$ [where $d(T_\rho)^*P_0/dP_0 = X$ and $d(T_\rho^{-1})^*P_0/dP_0 = Y$] it follows that if $Y(\omega)$ is of the Girsanov form, so is $X(\omega)$.

Consider now the classical Wiener space: $\omega = \{W_t,\ 0 \le t \le 1\}$, where W stands for the standard Wiener process. Let $\mathscr{G}_t$ denote the subsigma fields generated by $\{W_\theta,\ 0 \le \theta \le t\}$ and assume that $F(\omega)$ satisfies the requirements of Theorem 6.4 of [2]. Writing in this case $F(\omega) = \{F_t(\omega),\ 0 \le t \le 1\}$, further assume that $F_t(\omega)$ is $\mathscr{G}_t$-measurable; then obviously $\det_2(I_H + \rho\nabla F) = 1$ follows from a comparison of the Ramer and Girsanov formulas. Two direct proofs of this result, based on the properties of $\det_2$ (without any appeal to the Girsanov-type results) will now be pointed out. Note first that since we are dealing with the classical Wiener case we have the representations $F_t = \int_0^t f_s\, ds$ and

$$\langle \nabla F_t, h\rangle_H = \int_0^1 D_s f_t \cdot \dot{h}(s)\, ds,$$

where $D_s f_t$ is a Hilbert Schmidt kernel on $[0,1]^2$ and, since F_t is $\mathscr{G}_t$-measurable, $D_s f_t = 0$ whenever $s > t$. In general, for the classical Wiener case, by the Hilbert–Fredholm formula (cf., e.g., Theorem 9.4 of [5]):

$$\det_2(I_H + \rho\nabla F) = 1 + \sum_{m=1}^{\infty} \frac{\rho^m}{m} \int_{[0,1]^m} \det\big(K(s_i, s_j)\big)_{m\times m}\, ds_1 \dots ds_m,$$

where $(K(s_i, s_j))_{m\times m}$ is an $m \times m$ matrix with the (i,j)th entry given by 0 if $i = k$ and by $D_{s_i} f_{s_j}$ for $j \ne i$. In our case, since $D_s f_t = 0$ for $s > t$, $(K(s_i, s_j))_{m\times m}$ is a triangular matrix with zeros on the diagonal and its determinant is zero, consequently, $\det_2(I_H + \rho\nabla F) = 1$. The second proof of this result is based on the result of this note: Note that for any $K(s,t)$, $L(s,t)$ which are square integrable on $[0,1]^2$ and for any complete orthonormal system on $[0,1]$, $\{\phi_i(t), i = 1, 2, \dots\}$,

$$\mathrm{Trace}(K, L) = \sum_i \int_0^1 \left(\int_0^1 K(s,\theta)\phi_i(s)\, ds\right) \cdot \left(\int_0^1 L(\theta,t)\phi_i(t)\, dt\right) d\theta.$$

Hence, by the Parseval theorem,

$$\mathrm{Trace}(K, L) = \int_{[0,1]^2} K(s, \theta) L(\theta, s)\, d\theta\, ds.$$

Now, if $K(s, t) = 0$ and $L(s, t) = 0$ for $s > t$, then $\mathrm{Trace}(K, L) = 0$, and moreover $M(s, t) = 0$ for $s > t$, where

$$M(s, t) = \int_0^1 K(s, \theta) L(\theta, t)\, d\theta.$$

Consequently, if F_t is $\mathscr{G}_t$-measurable, then $D_s f_t$ satisfies condition (b) of the theorem a.s.-P_0.

We note that in the abstract Wiener space setup, if ∇F is (a.s.-P_0) both quasinilpotent and of trace class, then Trace $\nabla F = 0$ and the Fredholm determinant coincides with the Carleman–Fredholm determinant. In the classical Wiener space, the class of Ogawa integrable integrands $\{u_s, 0 \le s \le 1\}$ (cf. [3]) is larger than the class of integrands for which $D_t u_s$ is of trace class. *If, however,* $D_t u_s$ *is both quasinilpotent and of trace class, then the Skorohod integral coincides with the Ogawa integral.*

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Department of Electrical Engineering
Technion—Israel Institute of Technology
Haifa 32000
Israel