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Probabilités/*Probability Theory*

# Compact families of Wiener functionals

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***Abstract*** – Integral criteriums of Ascoli type.

## Familles compactes de fonctionnelles de Wiener

***Résumé*** – *Critères intégraux du type d'Ascoli.*

***Version française abrégée*** – Les espaces de Sobolev de fonctionnelles de Wiener sont définis par les relations (1) et (2).

THÉORÈME 1. – *La partie* $\mathscr{G}$ *définie par* (3) (*où* C *est un opérateur compact*) *est compacte dans* $L^2(\Omega)$.

*Preuve.* – On choisit une base diagonalisant C. Le produit des polynômes de Hermite construits sur cette base donnent les développements (4) et (5). Alors (6) et (7) contiennent la démonstration.

PROPOSITION 1. – *Étant donné une partie compacte* $\mathscr{G}$ *d'un espace* $L^2(\Omega, H_0)$, *on peut trouver un opérateur compact* C *tel que* (8) *soit vérifiée.*

*Preuve.* – D'après le théorème de Dini, $\rho_n$ [définie en (9)] tend uniformément vers zéro. La construction de C est alors donnée par (10).

THÉORÈME 2. – *Un opérateur de* $D_\infty^2(\Omega)$ *est relativement compact si et seulement si* (10) *est satisfaite.*

THÉORÈME 3. – *Une partie de* $D_\infty(\Omega)$ *est relativement compacte si et seulement si elle est bornée dans* $D_r^p(\Omega)$ (*quels que soient* $p$ *et* $r$) *et si* (11) *est vérifiée.*

APPLICATIONS. – A. Compacité de la boule unité d'un espace de Sobolev hilbertien dans $L^2$.

B. Compacité dans $L^2$ des fonctionnelles de Wiener associées à une équation différentielle stochastique à coefficients lipschitziens.

La démonstration de B fait intervenir un opérateur de dérivée fractionnaire [*cf.* (13)].

C. Compacité dans $D^\infty$ des fonctionnelles de Ito associées à une équation différentielle stochastique à coefficients $C^\infty$ bornés.

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Let $\{(\Omega, \mathscr{A}, P); H\}$ be a Gaussian probability space. Namely $(\Omega, \mathscr{A}, P)$ is a probability space and H is a separable closed subspace of $L^2(\Omega)$ formed by Gaussian random variables such that H generates the $\sigma$-field $\mathscr{A}$.

We denote by D the derivative operator acting on elementary smooth random variables as follows

$$D(f(h_1, \ldots, h_n)) = \sum_{i=1}^{n} \partial_i f(h_1, \ldots, h_n) h_i$$

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where $h_i \in \mathrm{H}$ and $f \in \mathrm{C}_b^\infty(\mathbf{R}^n)$. For every $p>1$ and $r \in \mathbf{N}$ we denote by $\mathrm{D}_r^p$ the closure of the family of elementary smooth random variables with respect to the norm

$$(1) \qquad \|\mathrm{F}\|_{p,r} = \sum_{i=0}^{r} \|\mathrm{D}^i \mathrm{F}\|_{\mathrm{L}^p(\Omega, \mathrm{H}^{\otimes i})}$$

where $\mathrm{D}^i$ denotes the $i$-th iteration of D. Set

$$(2) \qquad \mathrm{D}_\infty^p = \bigcap_{r \geqq 0} \mathrm{D}^{p,r} \quad \text{and} \quad \mathrm{D}_\infty = \bigcap_{p,r} \mathrm{D}^{p,r}.$$

The aim of this Note is to characterize compact subsets in the spaces $\mathrm{D}_\infty^2$ and $\mathrm{D}_\infty$. The first result in this direction is the following.

THEOREM 1. — *Let* C *be a selfadjoint compact operator on* H *with dense image. Then for any* $c>0$ *the set*

$$(3) \qquad \mathcal{G} = \{ \mathrm{G} \in \mathrm{D}_1^2 : \|\mathrm{G}\|_{\mathrm{L}^2(\Omega)} + \|\mathrm{C}^{-1}\mathrm{DG}\|_{\mathrm{L}^2(\Omega;\mathrm{H})} \leqq c \}$$

*is relatively compact in* $\mathrm{L}^2(\Omega)$.

*Proof.* — Consider a complete orthonormal system of elements $\{e_i, i \geqq 1\}$ in H such that $\mathrm{C}e_i = \alpha_i e_i$, $\alpha_i > 0$. The compacity of C implies $\lim_{i \to \infty} \alpha_i = 0$.

We denote by $\Lambda$ the family of all sequences of nonnegative integers $q=(q_1, q_2, \ldots)$ such that $q_i = 0$ except for a finite number of indices. For any $q \in \Lambda$ we write $q! = \prod_{i=1}^{\infty} q_i!$ Let $\mathrm{H}_m(x)$ be the $m$-th Hermite polynomial normalized in such a way that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \mathrm{H}_m^2(x) \exp\left(-\frac{x^2}{2}\right) dx = \frac{1}{m!}.$$

Set $\mathrm{H}_q(\omega) = \prod_{i=1}^{\infty} \mathrm{H}_{q_i}(e_i(\omega))$ for any $q \in \Lambda$. Then $\{\mathrm{H}_q, q \in \Lambda\}$ is a basis of $\mathrm{L}^2(\Omega)$.

Let $\mathrm{G} \in \mathrm{D}_1^2$ be a random variable such that

$$\|\mathrm{G}\|_{\mathrm{L}^2(\Omega)} + \|\mathrm{C}^{-1}\mathrm{DG}\|_{\mathrm{L}^2(\Omega;\mathrm{H})} \leqq c.$$

We have

$$(4) \qquad \mathrm{G} = \sum_{q \in \Lambda} \lambda_q \mathrm{H}_q,$$

and

$$(5) \qquad \mathrm{DG} = \sum_{q \in \Lambda} \sum_{i : q_i > 0} \lambda_q \mathrm{H}_{q_i - 1}(e_i) \prod_{j \neq i} \mathrm{H}_{q_j}(e_j) e_i.$$

From these expressions we obtain

$$\|\mathrm{G}\|_{\mathrm{L}^2(\Omega)}^2 = \sum_{q \in \Lambda} \frac{1}{q!} \lambda_q^2,$$

and

$$\|\mathrm{C}^{-1}\mathrm{DG}\|_{\mathrm{L}^2(\Omega;\mathrm{H})}^2 = \sum_i \frac{1}{\alpha_i^2} \int_\Omega \Big| \sum_{q, q_i \leqq 0} \lambda_q \mathrm{H}_{q_i - 1}(e_i) \prod_{j \neq i} \mathrm{H}_{q_j}(e_j) \Big|^2 d\mathrm{P}$$

$$= \sum_i \frac{1}{\alpha_i^2} \sum_{q : q_i > 0} \lambda_q^2 \frac{1}{(q_i - 1)! \prod_{j \neq i} q_j!}.$$

Therefore

$$(6) \qquad \|\mathrm{C}^{-1}\mathrm{DG}\|_{\mathrm{L}^2(\Omega;\mathrm{H})}^2 = \sum_{q \in \Lambda} \frac{1}{q!} \lambda_q^2 \left( \frac{1}{\alpha^2} \cdot q \right),$$

where $(1/\alpha^2) \cdot q = \sum_{i=1}^{\infty} (1/\alpha_i^2) q_i$.

Notice that when $\alpha_i=1$ for all $i$ (then $C=I$ is not compact) we get the factor $\sum_{i=q}^{\infty} q_i$ which is equal to the order of the Wiener chaos.

Fix $R>0$ and define

$$A_R=\left\{q\in\Lambda:\frac{1}{\alpha^2}.q<R\right\}.$$

Then the set $A_R$ is finite and

$$\|G\|^2_{L^2(\Omega)}\sum_{q\neq A_R}\frac{1}{q!}\lambda_q^2\leqq\|C^{-1}DG\|^2_{L^2(\Omega;H)}+\sum_{q\in A_R}\frac{1}{q!}\lambda_q^2. \tag{7}$$

From this we deduce the relative compactness of $\mathscr{G}$. Indeed, for any $\varepsilon>0$ we can find a finite number of elements $\{\lambda_q^j: q\in A_R: 1\leqq j\leqq j_0(R,\varepsilon)\}$ such that

$$\sup_{G\in\mathscr{G}}\inf_j\left(\sum_{q\in A_R}\frac{|\lambda_q-\lambda_q^j|^2}{q!}\right)<\frac{\varepsilon}{2}.$$

Set $G^j=\sum_{q\in A_R}\lambda_q^j H_q$. Then the $L^2(\Omega)$-balls of center $G^j$, $1\leqq j\leqq j_0(R,\varepsilon)$ and radius $\varepsilon>0$ cover the set $\mathscr{G}$ provided $R<\varepsilon/2c^2$, because $\inf_j\|G-G^j\|^2_{L^2(\Omega)}\leqq(c^2/R)+(\varepsilon/2)<\varepsilon$. ■

Theorem 1 provides a sufficient condition for relative compactness on $L^2(\Omega)$. From this condition we will get necessary and sufficient conditions for relative compactness in the spaces $D^2_\infty$ and $D_\infty$. In order to establish the necessity part we will need the following result.

Proposition 1. — *Let* $(\Omega_0, \mathscr{A}_0, P_0)$ *be an arbitrary probability space and let* $H_0$ *be a real separable Hilbert space. Let* $\mathscr{G}$ *be a compact subset of* $L^2(\Omega_0; H_0)$. *Then there exists a selfadjoint compact operator* C *on* $H_0$ *with dense image such that*

$$\sup_{F\in\mathscr{G}} E(\|C^{-1}F\|^2_{H_0})<\infty. \tag{8}$$

*Proof.* — Let $\{e_i, i\geqq 1\}$ be a complete orthonormal system on $H_0$ and for any $n\geqq 1$ let $V_n$ be the finite dimensional subspace spanned by $e_1,\ldots,e_n$. We will denote by $P_n$ the orthogonal projection on $V_n$. We have

$$\rho_n(F)=:E(\|F-P_nF\|^2)\downarrow 0 \tag{9}$$

as $n$ tend to infinity. The set $\mathscr{G}$ being compact we deduce by Dini's Theorem that

$$\lim_{n\to\infty}\sup_{F\in\mathscr{G}}\rho_n(F)=0.$$

Consequently we can find an increasing sequence $\{s(k), k\geqq 1\}$ such that $\rho_{s(k)}(F)<(1/k^2)$ for all $F\in\mathscr{G}$. Then we define a compact operator C by

$$C=\sum_{k=1}^{\infty}k^{-1/2}[P_{s(k)}[P_{s(k)}\ominus P_{s(k-1)}], \qquad P_{s(0)}=:0. \tag{10}$$

We have, for any $F\in\mathscr{G}$

$$E(\|C^{-1}F\|^2_{H_0})=\sum_{k=1}^{\infty}k\|(P_{s(k)}\ominus P_{s(k-1)})(F)\|^2_{L^2(\Omega_0;H_0)}$$

$$=\sum_{N=1}^{\infty}\sum_{k=N}^{\infty}\|(P_{s(k)}\ominus P_{s(k-1)})(F)\|^2_{L^2(\Omega_0;H_0)}\leqq\sum_{N=1}^{\infty}\|P_{s(N)}(F)\|^2\leqq\sum_{N=1}^{\infty}\frac{1}{N^2},$$

which completes the proof. ■

Combining Proposition 1 and Theorem 1 we deduce the following results:

THEOREM 2. — *A subset* $\mathscr{G}$ *of* $D^2_\infty$ *is relatively compact if and only if it is bounded in* $L^2(\Omega)$ *and for all* $n \geqq 1$ *there exists a selfadjoint compact operator* $C_n$ *on* $H^{\otimes n}$ *such that*

$$\sup_{F \in \mathscr{G}} E(\| C_n^{-1} D^n F \|^2_{H^{\otimes n}}) < \infty. \tag{11}$$

THEOREM 3. — *A subset* $\mathscr{G}$ *of* $D_\infty$ *is relatively compact if and only if the following two conditions are satisfied:*

(i) $\sup_{F \in \mathscr{G}} \| F \|_{p,r} < \infty$ *for all* $p$, $r$.

(ii) *For all* $n \geqq 1$ *there exists a selfadjoint compact operator* $C_n$ *on* $H^{\otimes n}$ *such that*

$$\sup_{F \in \mathscr{G}} E(\| C_n^{-1} D^n F \|^2_{H^{\otimes n}}) < \infty.$$

APPLICATIONS A. — Let $\mu$ be a Gaussian centered measure on a separable Hilbert space $H_1$, such that $\int_H \| x \|^2 \mu(dx) < \infty$. Then for any $c > 0$ the set

$$\{ f \in C^1(H) : \| f \|_{L^2(\mu)} + \| \nabla f \|_{L^2(\mu : H_1)} \leqq c \}$$

is relatively compact on $L^2(\mu)$. This follows from Theorem 1 taking into account that the injection $i : H_1 \to L^2(\mu)$ given by $i(h) = \langle h, . \rangle_H$ is a compact operator. In this case the Gaussian space H is $\overline{i(H_1)}^{L^2(\mu)}$.

APPLICATION B. — For each $c > 0$ we will denote by $\mathscr{H}_c$ the family of all Lipschitz functions $\varphi : \mathbf{R}^d \to \mathbf{R}$ such that $|\varphi(0)| \leqq c$ and the Lipschitz constant of $\varphi$ is bounded by $c$. Let $\{ W(t), t \geqq 0 \}$ be a $k$-dimensional Brownian motion defined on the canonical probability space $(\Omega, \mathscr{F}, P)$.

Consider the solution $\{ X_t, 0 \leqq t \leqq 1 \}$ of the stochastic differential equation

$$\left\{ \begin{aligned} dX_t &= \sum_{i=1}^{k} \sigma_i(X_t) \, dW_t^i + b(X_t) \, dt \\ X_0 &= x_0, \end{aligned} \right. \tag{12}$$

where $x_0 \in \mathbf{R}^d$.

PROPOSITION 2. — *The random variables* $X_1$ *obtained by choosing* $\sigma_i$, $b$ *in* $\mathscr{H}_c$, $1 \leqq i \leqq k$ *and* $|x_0| \leqq c$ *is relatively compact in* $L^2(\Omega; \mathbf{R}^d)$.

In order to show this proposition we need some preliminary material. Let $\{ v_s, s \geqq 0 \}$ be the Haar basis of $L^2([0, 1])$. Namely $v_0 = 1$ and if $s = 2^k + j$, $k \geqq 0$, $0 \leqq j \leqq 2^k$ then

$$v_s(r) = 2^{k/2} [1_{[j 2^{-k}, (2j+1) 2^{-k-1}]} - 1_{[(2j+1) 2^{-k-1}, (j+1) 2^{-k}]}]$$

For any $0 < \alpha < 1/2$ we define the operator $A_\alpha$ on $L^2([0, 1])$ by

$$A_\alpha v_s = 2^{k\alpha} v_s, \qquad \text{if} \quad s = 2^k + j$$

and

$$A_\alpha 1 = 1.$$

LEMMA 1. — *For every* $\beta$ *such that* $\alpha < \beta < (1/2)$, *there exists a constant* $c_1$ *such that*

$$\| A_\alpha f \|_{L^2([0,1])} \leqq c_1 \left\{ \| f \|_{L^2([0,1])} + \left( \int_0^1 \int_0^1 \frac{| f(t) - f(t') |^2}{| t - t' |^{1+2\beta}} \, dt \, dt' \right)^{1/2} \right\}. \tag{13}$$

*Proof.* — Let $f \in L^2([0, 1])$ be a function with the expansion $f = \sum_{s \geqq 0} \langle f, v_s \rangle v_s$. Then if $f_k = \sum_{0 \leqq s < 2^k} \langle f, v_s \rangle v_s$ for $k \geqq 1$ and $f_0 = \langle f, v_0 \rangle$, we obtain

$$(14)\quad \| A_\alpha f \|_{L^2([0, 1])} = \left( \| A_\alpha f_0 \|^2_{L^2([0, 1])} + \left\| A_\alpha \sum_{k=0}^{\infty} (f_{k-1} - f_k) \right\|^2_{L^2([0, 1])} \right)^{1/2} \leqq \| f \|_{L^2([0, 1])} + \left( \sum_{k=0}^{\infty} \| A_\alpha (f_{k-1} - f_k) \|^2_{L^2([0, 1])} \right)^{1/2}.$$

Set $A_k = \bigcup_{0 \leqq j \leqq 2^k} [j 2^{-k}, (2j+1) 2^{-k-1}]$. Then

$$(15)\quad \| A_\alpha (f_{k-1} - f_k) \|^2_{L^2([0, 1])} = 2^{2(\alpha+1)k} \left| \int_{A_k \times A_k^c} (f(t) - f(t'))\, dt\, dt' \right|^2 \leqq 2^{2(\alpha+1)k} \int_{A_k \times A_k^c} | f(t) - f(t') |^2\, dt\, dt' \leqq 2^{-(\beta-\alpha)2k} \int_0^1 \int_0^1 \frac{| f(t) - f(t') |^2}{| t - t' |^{1+2\beta}}\, dt\, dt'.$$

Then the estimates (14), (15) prove the desired inequality. ■

*Proof of Proposition* 2. — From Theorem 1 it suffices to show that

$$\sup_{| x_0 | \leqq c,\, \sigma_i,\, b \in \mathscr{H}_c} \{ E(| X_1 |^2) + E(\| A_\alpha D X_1 \|^2_{L^2([0, 1]; \mathbf{R}^d)}) \} < \infty$$

where $A_\alpha$ is the operator defined before. The derivative $D_s X_t$, $0 \leqq s \leqq t \leqq 1$ is the solution of the linear stochastic equation

$$D_{s, j} X_t = \sigma_j (X_s) + \sum_{i=1}^{k} \int_s^t A_i (u) . D_{s, j} X_u\, dW_u^i + \int_s^t B(u) D_{s, j} X_u\, du,$$

where $A_i$ and B are adapted matrix valued processes uniformly bounded by $c$ (if the coefficients are differentiable $A_i(u) = \nabla \sigma_i (X_u)$ and $B(u) = \nabla b (X_u)$). Consequently we have, for any $\varepsilon > 0$,

$$E[| D_s X_1 - D_{s'} X_1 |^2] \leqq c_2 | s - s' |^{1-\varepsilon},$$

and Lemma 1 allows to conclude. ■

In a similar way we can prove the following result:

Application C. — Fix a sequence of positive numbers $c = \{ c_n, n \geqq 0 \}$, and $a > 0$. Let $\mathscr{H}^c_\infty$ be the set of all infinitely differentiable functions $\varphi : \mathbf{R}^d \to \mathbf{R}$ such that for each $n \geqq 0$ the $n$-th partial derivatives of $\varphi$ are bounded by $c_n$. Then the family of random variables, $X_1$ obtained from the equation (12) by choosing $\sigma_i$, and $b$ in $\mathscr{H}^c_\infty$ and with $| x_0 | \leqq a$ is relatively compact in $D_\infty (\mathbf{R}^d)$.

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