Chapter 4
Fractional Brownian motion

We now introduce the fractional Brownian motion. For notations, we refer to the end of this chapter where all the necessary notions of fractional deterministic calculus.

4.1 Wiener space for the fractional Brownian motion

Definition 4.1. For any H in $(0,1)$, the fractional Brownian motion of index (Hurst parameter) H, $\{B_H(t);\ t \in [0,1]\}$ is the centered Gaussian process whose covariance kernel is given by

$$R_H(s,t) = \mathbf{E}\left[B_H(s)B_H(t)\right] = \frac{V_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

where

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

Theorem 4.1. *Let $H \in (0,1)$, the sample-paths of W^H are Hölder continuous of any order less than H (and no more) and belong to $W_{\alpha,p}$ for any $p \geq 1$ and any $\alpha \in (0,H)$.*

Proof. Since, for any $\alpha \geq 0$, we have

$$\mathbf{E}\left[|B_H(t) - B_H(s)|^{\alpha}\right] = C_\alpha |t-s|^{H\alpha},$$

we have

$$\mathbf{E}\left[\iint_{[0,1]^2} \frac{|B_H(t) - B_H(s)|^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\right] = C_\alpha \iint_{[0,1]^2} |t-s|^{-1+p(H-\alpha)}\,\mathrm{d}t\,\mathrm{d}s.$$

This integral is finite as soon as $\alpha < H$ hence B_H belongs to $W_{\alpha,p}$ for any $\alpha < H$, any $p \geq 1$. Choose p arbitrary large and conclude that the sample-paths are Hölder continuous of any order less than H.

As a consequence of the results in [Arc95], we have

$$\mu_H \left(\limsup_{u \to 0^+} \frac{B_H(u)}{u^H \sqrt{\log \log u^{-1}}} = \sqrt{V_H} \right) = 1.$$

Hence it is impossible for B_H to have sample-paths Hölder continuous of an order greater than H.

The difference of regularity is evident on simulations of sample-paths, see Figure 4.1.

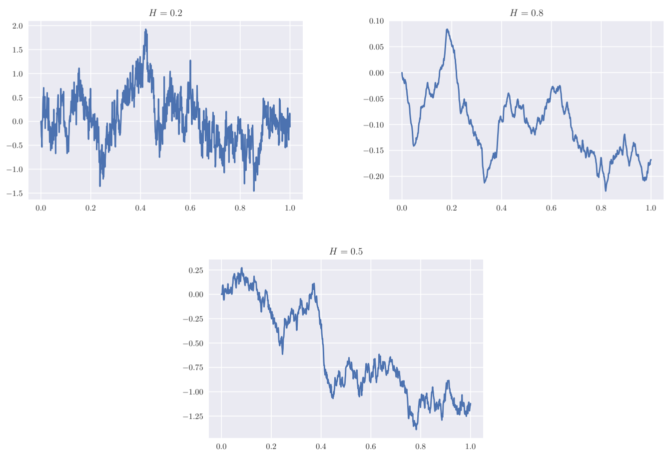


Fig. 4.1 Sample-path example for $H = 0.2$, $H = 0.5$ and $H = 0.8$.

As a consequence, B_H cannot be a semi-martingale as its quadratic variation is either null or infinite.

Theorem 4.2. *We have the following almost-sure limits:*

$$\lim_{n \to \infty} \sum_{j=1}^{n} \left| B_H\left(\frac{j}{n}\right) - B_H\left(\frac{j-1}{n}\right) \right|^2 = \begin{cases} 0 & \text{if } H > 1/2 \\ \infty & \text{if } H < 1/2. \end{cases}$$

Proof. Because of the specific form of the covariance kernel, it is easy to see that the process $(a^{-H} B_H(at), t \geq 0)$ has the same covariance kernel as B_H does, so that they do have the same distribution. This entails that

$$\sum_{j=1}^{n} \left| B_H\left(\frac{j}{n}\right) - B_H\left(\frac{j-1}{n}\right) \right|^H$$

has the same distribution as

$$\frac{1}{n}\sum_{j=1}^{n}\Big|B_H\Big(j\Big)-B_H\Big(j-1\Big)\Big|^H.$$

The ergodic theorem entails that this converges in L^1 and almost-surely to $\mathbf{E}\left[|B_H(1)|^H\right]$. Hence the result.

The next step is to describe the Cameron-Martin space attached to the fBm of index H. The general theory of Gaussian processes says that we must consider the self-reproducing Hilbert space defined by the covariance kernel, see the appendix of Chapter 1.

Definition 4.2. Let

$$\mathcal{H}^0=\operatorname{span}\{R_H(t,.),\ t\in[0,1]\},$$

equipped with the scalar product

$$\langle R_H(t,.),\ R_H(s,.)\rangle_{\mathcal{H}_H}=R_H(t,s). \tag{4.1}$$

The Cameron-Martin space of the fBm of Hurst index H, denoted by $\mathcal{H}_H$, is the completion of $\mathcal{H}^0$ for the scalar product defined in (4.1).

This is not a very practical definition but we can have a much better description of $\mathcal{H}_H$ thanks to the next theorems.

Theorem 4.3 (cf [SKM93, page 187]). *For $H\in(0,1)$, consider the function*

$$\begin{aligned}K_H\ :\ [0,1]^2 &\longrightarrow \mathbf{R}\\ (t,s) &\longmapsto \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}\,F\big(H-1/2,1/2-H,H+1/2,1-t/s\big)\end{aligned} \tag{4.2}$$

and the integral transform of kernel K_H, i.e.

$$\begin{aligned}K_H\ :\ L^2([0,1]\to\mathbf{R};\ \lambda) &\longrightarrow I_{H+1/2,2}\\ f &\longmapsto \left(t\mapsto\int_0^t K_H(t,s)f(s)\ ds\right).\end{aligned}$$

The map K_H is an isomorphism from $L^2([0,1]\to\mathbf{R};\ \lambda)$ onto $I_{H+1/2,2}$ and

$$\begin{aligned}K_Hf &= I_{0^+}^{2H}x^{1/2-H}I_{0^+}^{1/2-H}x^{H-1/2}f \quad \text{for } H\le 1/2,\\ K_Hf &= I_{0^+}^{1}x^{H-1/2}I_{0^+}^{H-1/2}x^{1/2-H}f \quad \text{for } H\ge 1/2.\end{aligned}$$

Note that if $H\ge 1/2$, $r\to K_H(t,r)$ is continuous on $(0,t]$ so that we can include t in the indicator function.

Theorem 4.4. *For any $H \in (0,1)$, $R_H(s,t)$ can be written as*

$$R_H(s,t) = \int_0^1 K_H(s,r)K_H(t,r)\ dr. \tag{4.3}$$

If we identify integral operators and their kernel, this amounts to say that

$$R_H = K_H \circ K_H^*.$$

Proof. For $H > 1/2$, it is easy to see that

$$R_H(s,t) = \frac{V_H}{4H(2H-1)} \int_0^t \int_0^s |r-u|^{2H-2}\ \mathrm{d}u\ \mathrm{d}r$$

Moreover (see [BVP88]),

$$\frac{V_H}{4H(2H-1)}|r-u|^{2H-2} = (ru)^{H-1/2} \int_0^{r\wedge u} v^{1/2-H}(r-v)^{H-3/2}(u-v)^{H-3/2}\ \mathrm{d}v.$$

Hence for $H > 1/2$, (4.3) holds with

$$K_H(t,r) = \frac{r^{1/2-H}}{\Gamma(H-1/2)} \int_r^t u^{H-1/2}(u-r)^{H-3/2}\ du\ \mathbf{1}_{[0,t]}(r).$$

A change of variable in this equation transforms the integral term in

$$(t-r)^{H-1/2} r^{H-1/2} \int_0^1 u^{H-3/2} \left(1-(1-t/r)u\right)^{H-1/2}\ du.$$

By the definition (4.19) of hypergeometric functions, we see that (4.2) holds true for $H > 1/2$. Using property (4.21), we have

$$K_H(t,r) = \frac{2^{-2H}\sqrt{\pi}}{\Gamma(H)\sin(\pi H)} r^{H-1/2} + \frac{1}{2\Gamma(H+1/2)}(t-r)^{H-1/2} F(1/2-H, 1, 2-2H, \frac{r}{t}).$$

If $H < 1/2$ then the hypergeometric function of the latter equation is continuous with respect to r on $[0,t]$ because $2-2H-1-1/2+H = 1/2-H$ is positive. Hence, for $H < 1/2$, $K_H(t,r)(t-r)^{1/2-H}r^{1/2-H}$ is continuous with respect to r on $[0,t]$. For $H > 1/2$, the hypergeometric function is no more continuous in t but we have [NU88] :

$$F(1/2-H,1,2-2H,\frac{r}{t}) = C_1F(1/2-H,1,H+1/2,1-r/t) \\ + C_2(1-r/t)^{1/2-H}(r/t)^{2H-1}.$$

Hence, for $H \geq 1/2$, $K_H(t,r)r^{H-1/2}$ is continuous with respect to r on $[0,t]$. Fix $\delta \in [0,1/2)$ and $t \in (0,1]$, we have :

$$|K_H(t,r)| \leq Cr^{-|H-1/2|}(t-r)^{-(1/2-H)_+}\mathbf{1}_{[0,t]}(r)$$

where C is uniform with respect to $H \in [1/2-\delta, 1/2+\delta]$. Thus, the two functions defined on $\{H \in \mathcal{C},\ |H-1/2| < 1/2\}$ by

$$H \in (0,1) \longmapsto R_H(s,t) \text{ and } H \in (0,1) \longmapsto \int_0^1 K_H(s,r)K_H(t,r)\ \mathrm{d}r$$

are well defined, analytic with respect to H and coincide on $[1/2,1)$, thus they are equal for any $H \in (0,1)$ and any s and t in $[0,1]$.

In the previous proof we proved a result which is so useful in its own that it deserves to be a theorem :

Theorem 4.5. *For any* $H \in (0,1)$, *for any* t, *the function*

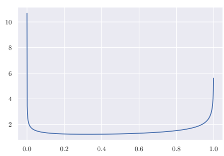
$$\begin{aligned}[0,t] &\longrightarrow \mathbf{R}\\ r &\longmapsto K_H(t,r)r^{|H-1/2|}(t-r)^{(1/2-H)_+}\end{aligned}$$

is continuous on $[0,t]$.

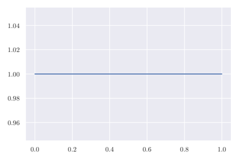
Moreover, there exists a constant c_H *such for any* $0 \leq r \leq t \leq 1$

$$|K_H(t,r)| \leq c_H\, r^{-|H-1/2|}(t-r)^{-(1/2-H)_+}. \tag{4.4}$$

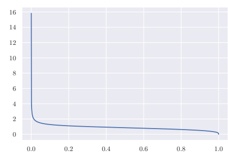
These continuity results are illustrated by the following pictures.



$K_{0.25}(1,s)$



$K_{0.5}(1,s)$



$K_{0.75}(1,s)$

Remark 4.1. We already know that the fBm is all the more regular than its Hurst index is close to 1. However, we see that the kernel K_H is more and more singular when H goes to 1. This means that it is probably a bad idea to devise properties of B_H using the properties of K_H. On the other hand, as an operator K_H is more and more regular as H increases. This indicates that the efficient approach is to work with K_H as an operator. We tried to illustrate this line of reasoning in the next results.

The structure of the Cameron-Martin space can now be fully described.

Theorem 4.6. *The Cameron-Martin of the fractional Brownian motion is* $\mathcal{H}_H = \{K_H \dot{h};\ \dot{h} \in L^2([0,1] \to \mathbf{R};\ \lambda)\}$, *i.e., any* $h \in \mathcal{H}_H$ *can be represented as*

$$h(t) = K_H \dot{h}(t) = \int_0^1 K_H(t,s)\dot{h}(s)\ ds,$$

where $\dot{h}$ *belongs to* $L^2([0,1] \to \mathbf{R};\ \lambda)$. *For any* $\mathcal{H}_H$*–valued random variable* u, *we hereafter denote by* $\dot{u}$ *the* $L^2([0,1] \to \mathbf{R};\ \lambda))$*-valued random variable such that*

$$u(w,t) = \int_0^t K_H(t,s)\dot{u}(w,s)\, ds.$$

The scalar product on $\mathcal{H}_H$ *is given by*

$$(h,g)_{\mathcal{H}_H} = (K_H\dot{h}, K_H\dot{g})_{\mathcal{H}_H} = (\dot{h},\dot{g})_{L^2([0,1]\to\mathbf{R};\ \lambda)}.$$

We can now construct the fractional Wiener measure as we did for the ordinary Brownian motion.

Theorem 4.7. *Let* $(\dot{h}_m,\ m \ge 0)$ *be a complete orthonormal basis of* $L^2([0,1] \to \mathbf{R};\ \lambda)$ *and* $h_m = K_H\dot{h}_m$. *Consider the sequence*

$$S_n^H(t) = \sum_{m=0}^{n} X_m h_m(t)$$

where $(X_m,\ m \ge 0)$ *is a sequence of independent standard Gaussian random variables. Then,* $(S_n^H,\ n \ge 0)$ *converges, with probability* 1, *in* $W_{\alpha,p}$ *for any* $\alpha < H$.

Proof. The proof proceeds exactly as the proof of Theorem 1.5. The trick is to note that

$$(h_m(t) - h_m(s))^2 = \langle K_H(t,.) - K_H(s,.),\ \dot{h}_m\rangle^2_{\mathcal{H}_H},$$

so that

$$\begin{aligned}\sum_{m=0}^{\infty}(h_m(t) - h_m(s))^2 &= \|K_H(t,.) - K_H(s,.)\|^2_{L^2([0,1]\to\mathbf{R};\ \lambda)} \\ &= R_H(t,t) - R_H(s,s) - 2R_H(t,s) = V_H|t-s|^{2H}.\end{aligned}$$

Moreover,

$$\int_{[0,1]^2} |t-s|^{pH-1-\alpha p}\ \mathrm{d}s\ \mathrm{d}t < \infty \text{ if and only if } \alpha < H.$$

This means, by dominated convergence, that

$$\sup_{n\ge M} \mathbf{E}\left[\|S_n^H - S_M^H\|_{W_{\alpha,p}}^p\right] = \iint_{[0,1]^2} \Big(\sum_{m=M+1}^{\infty} (h_m(t)-h_m(s))^2\Big)^{p/2} |t-s|^{-1-\alpha p}\, \mathrm{d}s\, \mathrm{d}t \xrightarrow{M\to\infty} 0,$$

provided that $\alpha < H$. The proof is finished as in Theorem 1.5.

Remark 4.2. Theorem 4.3 implies that as a vector space, $\mathcal{H}_H$ is equal to $I_{0^+}^{H+1/2}(L^2([0,1]))$ but the norm on each of these spaces are different since

$$\|K_H\dot{h}\|_{\mathcal{H}_H} = \|\dot{h}\|_{L^2([0,1]\to\mathbf{R};\,\lambda)}$$
$$\text{and } \|K_H\dot{h}\|_{I_{0^+}^{H+1/2}(L^2([0,1]))} = \|(I_{0^+}^{-H-1/2}\circ K_H)\dot{h}\|_{L^2([0,1]\to\mathbf{R};\,\lambda)}.$$

In what follows, W may be taken either as $\mathcal{C}_0([0,1],\mathbf{R})$ or as any of the spaces $W_{\gamma,p}$ with

$$p\ge 1,\ 0<\gamma<H.$$

For any $H\in(0,1)$, μ_H is the unique probability measure on W such that the canonical process $(B_H(s);\ s\in[0,1])$ is a centered Gaussian process with covariance kernel R_H :

$$\mathbf{E}_H[B_H(s)B_H(t)] = R_H(s,t).$$

The canonical filtration is given by $\mathcal{F}_t^H = \sigma\{W_s,\ s\le t\}\vee\mathcal{N}_H$ and $\mathcal{N}_H$ is the set of the μ_H–negligible events. The analog of the diagram 1.1 reads as

$$\begin{array}{ccccc} \mathcal{W}^* & \xrightarrow{\mathfrak{e}^*} & \mathcal{H}_H{}^* = (I_{H+1/2,2})^* & & \\ & & \big\downarrow\simeq & & \\ L^2 & \xrightarrow{K_H} & \mathcal{H}_H = I_{H+1/2,2} & \xrightarrow{\mathfrak{e}} & \mathcal{W} \end{array}$$

Fig. 4.2 Embeddings and identification for fractional Brownian motion.

We can as before, search for the image of ε_t by $\mathfrak{e}^*$. We have, for $h\in\mathcal{H}_H$, on the one hand,

$$h(t) = \langle \varepsilon_t, \mathfrak{e}(h)\rangle_{\mathcal{W}^*,\mathcal{W}} = \langle \mathfrak{e}^*(\epsilon_t),\, h\rangle_{\mathcal{H}_H}.$$

On the other hand,

$$h(t) = K_H\dot{h}(t) = \langle K_H(t,.),\, \dot{h}\rangle_{L^2([0,1]\to\mathbf{R};\,\lambda)} = \langle R_H(t,.), h\rangle_H.$$

Hence,

$$\mathfrak{e}^*(\varepsilon_t) = R_H(t,.) \text{ and } K_H^{-1}(\mathfrak{e}^*(\varepsilon_t)) = K_H(t,.).$$

Recall that for the ordinary Brownian motion, we have

$$\mathfrak{e}^*(\varepsilon_t) = t \wedge . = R_{1/2}(t,.) \text{ and } K_H^{-1}(\mathfrak{e}^*(\varepsilon_t)) = \mathbf{1}_{[0,t]}(.) = K_{1/2}(t,.).$$

Theorem 4.8. *For any z in $\mathcal{W}^*$,*

$$\int_{\mathcal{W}} e^{i\langle z,\omega\rangle_{\mathcal{W}^*,\mathcal{W}}} \ d\mu_H(\omega) = \exp\Big(-\frac{1}{2}\|\mathfrak{e}^*(z)\|^2_{\mathcal{H}_H}\Big). \tag{4.5}$$

Proof. By dominated convergence, we have

$$\begin{aligned}\int_{\mathcal{W}} e^{i\langle z,\omega\rangle_{\mathcal{W}^*,\mathcal{W}}} \ \mathrm{d}\mu_H(\omega) &= \lim_{n\to\infty} \mathbf{E}\left[\exp\left(i\sum_{m=0}^{n} X_m \left\langle z,\, \mathfrak{e}(K_H\dot{h}_m)\right\rangle_{\mathcal{W}^*,\mathcal{W}}\right)\right]\\ &= \lim_{n\to\infty} \exp\left(-\frac{1}{2}\sum_{m=0}^{n}\left\langle \mathfrak{e}^*(z),\, K_H\dot{h}_m\right\rangle^2_{\mathcal{H}}\right)\\ &= \exp\left(-\frac{1}{2}\sum_{m=0}^{\infty}\left\langle \mathfrak{e}^*(z),\, K_H\dot{h}_m\right\rangle^2_{\mathcal{H}}\right)\\ &= \exp\Big(-\frac{1}{2}\|\mathfrak{e}^*(z)\|^2_{\mathcal{H}_H}\Big),\end{aligned}$$

according to the Parseval identity.

The Wiener integral is constructed as before as the extension of the map

$$\begin{aligned}\delta_{B_H} \,:\, \mathcal{W}^* \subset I_{1,2} &\longrightarrow L^2(\mu_H)\\ z &\longmapsto \langle z, B_H\rangle_{\mathcal{W}^*,\mathcal{W}}\,.\end{aligned}$$

By construction of the Wiener measure, the random variable $\langle z, B_H\rangle_{\mathcal{W}^*,\mathcal{W}}$ is Gaussian with mean 0 and variance $\|R_H(z)\|^2_{\mathcal{H}_H}$. For $z = \varepsilon_t$, we have

$$B_H(t) = \langle \varepsilon_t, B_H\rangle_{\mathcal{W}^*,\mathcal{W}} = \delta_{B_H}\big(R_H(t,.)\big).$$

For the Brownian motion, it is often easier to work with elements of L^2 instead of their image by $K_{1/2}$, which belongs to $I_{1,2}$. If we try to mimick this approach for the fractional Brownian motion, we should write:

$$B_H(t) = \delta_{B_H}\big(R_H(t,.)\big) = \delta_{B_H}\big(K_H(K_H(t,.))\big) = \int_0^1 K_H(t,s)\ \delta B_H(s),$$

which has to be compared to

$$B(t) = W^{1/2}(t) = \int_0^1 \mathbf{1}_{[0,t]}(s)\ \mathrm{d}W^{1/2}(s),$$

where the integral is taken in the Itô sense. Remark that these two equations are coherent since $K_{1/2}(t,.) = \mathbf{1}_{[0,t]}$.

Lemma 4.1. *The process* $B = (\delta_{B_H}(K_H(\mathbf{1}_{[0,t]})), t \in [0,1])$ *is a standard Brownian motion. For* $u \in L^2([0,1] \to \mathbf{R};\, \lambda)$,

$$\int_0^1 u(s)\ dB(s) = \delta_{B_H}(K_H u). \tag{4.6}$$

In particular,

$$B_H(t) = \int_0^t K_H(t,s)\ dB(s). \tag{4.7}$$

Proof. It is a Gaussian process by the definition of the Wiener integral. We just have to verify that it has the correct covariance kernel. For, it suffices to see that $\|K_H(\mathbf{1}_{[0,t]})\|^2_{\mathcal{H}_H} = t$. But,

$$\|K_H(\mathbf{1}_{[0,t]})\|^2_{\mathcal{H}_H} = \|\mathbf{1}_{[0,t]}\|^2_{L^2([0,1]\to\mathbf{R};\,\lambda)} = t.$$

This means that (4.6) holds for $u = \mathbf{1}_{[0,t]}$, hence for all piecewise constant functions u and by density, for all $u \in L^2$.

Remark 4.3. Eqn. (4.7) is known as the Karuhnen-Loeve representation. We could have started by considering a process defined by the right-hand-side of (4.7) and called it fractional Brownian motion. Actually, (4.7) is a stronger result: It says that starting from an fBm, one can construct a Brownian motion on the same probability space such that the representation (4.7) holds.

The gradient is defined as for the usual Brownian motion. The only modification is the Cameron-Martin space.

Definition 4.3. A function F is said to be cylindrical if there exists an integer n, $f \in \text{Schwartz}(\mathbf{R}^n)$, the Schwartz space on $\mathbf{R}^n$, $(h_1, \cdots, h_n) \in {\mathcal{H}_H}^n$ such that

$$F(\omega) = f(\delta_{B_H} h_1, \cdots, \delta_{B_H} h_n).$$

The set of such functionals is denoted by $\mathcal{S}_{\mathcal{H}_H}$.

Definition 4.4. Let $F \in \mathcal{S}$, $h \in \mathcal{H}_H$, with $F(\omega) = f(\delta_{B_H} h_1, \cdots, \delta_{B_H} h_n)$. Set

$$\nabla F = \sum_{j=1}^n \partial_j F(\delta_{B_H} h_1, \cdots, \delta_{B_H} h_n)\ h_j,$$

so that

$$\langle \nabla F, h \rangle_{\mathcal{H}_H} = \sum_{j=1}^n \partial_j F(\delta_{B_H} h_1, \cdots, \delta_{B_H} h_n)\ \langle h_j,\, h \rangle_{\mathcal{H}_H}.$$

Example 4.1. This means that

$$\nabla f(B_H(t)) = f'(B_H(t))R_H(t,.)$$

and if we denote $\dot{\nabla} = K_H^{-1}\nabla$ (which corresponds for $H = 1/2$ to take the time derivative of the gradient), we get

$$\dot{\nabla}_s f(B_H(t)) = f'(B_H(t))K_H(t,s).$$

The following theorem is an easy consequence of the properties of the maps K_H.

Theorem 4.9. *The operator $\mathcal{K}_H = K_H \circ K_{1/2}^{-1}$ is continuous and invertible from $I_{\alpha,p}$ into $W_{\alpha+H-1/2,p}$, for any $\alpha > 0$.*

Formally, we have $B_H = K_H(\dot{B}) = K_H \circ K_{1/2}^{-1}(B)$ so we can expect that

Theorem 4.10. *For any H, we have*

$$B = \mathcal{K}_H^{-1}(B_H),\ \mu_H - a.s. \tag{4.8}$$

Since,with μ_H-probability 1,

$$B_H = \sum_{m=0}^{\infty} X_m K_H(\dot{h}_m),$$

we find that, with μ_H-probability 1,

$$B = \sum_{m=0}^{\infty} X_m I^1(\dot{h}_m),$$

where $(\dot{h}_m,\ m \geq 0)$ is a complete orthonormal basis of $L^2([0,1] \to \mathbf{R};\ \lambda)$.

Proof. To prove such an identity, it is necessary and sufficient to check that

$$\mathbf{E}\left[\psi \int_0^1 B(t)g(t)\ \mathrm{d}t\right] = \mathbf{E}\left[\psi \int_0^1 \mathcal{K}_H^{-1}(B_H)g(t)\ \mathrm{d}t\right] \tag{4.9}$$

for any $g \in L^2([0,1] \to \mathbf{R};\ \lambda)$ and any $\psi \in \mathcal{S}_H$. Indeed, $L^2([0,1] \to \mathbf{R};\ \lambda) \otimes \mathcal{S}_H$ is a dense subset of $L^2([0,1] \to \mathbf{R};\ \lambda) \otimes L^2(\mathcal{W} \to \mathbf{R};\ \mu_H) = L^2([0,1] \times \mathcal{W} \to \mathbf{R};\ \lambda \otimes \mu_H)$ and (4.9) entails that $B = \mathcal{K}_H^{-1}(B_H)\ \lambda \otimes \mu_H$-almost-surely. This means that for there exists $A \subset [0,1] \times \mathcal{W}$ such that

$$\int_{[0,1]\times\mathcal{W}} \mathbf{1}_A(s,\omega)\ \mathrm{d}s\ \mathrm{d}\mu_H(\omega) = 0,$$

and

$$B(\omega,s) = \mathcal{K}_H^{-1}(B_H)(\omega,s) \text{ for } (s,\omega) \notin A.$$

Hence, for any $s \in [0,1]$, the section of A at s fixed, i.e. $A_s = \{\omega,\ (s,\omega) \in A\}$, is a μ_H-negligeable set.

The sample-paths of B are known to be continuous and that of B_H belongs to $W_{H,p}$ for any $p \geq 1$. Hence, according to Theorem 4.9, $\mathcal{K}_H^{-1}(B_H)$ almost-surely belongs to $W_{1/2,p}$ for any $p \geq 1$. Choose $p > 2$ so that $W_{1/2,p} \subset \mathrm{Hol}(1/2 - 1/p)$ to conclude that $\mathcal{K}_H^{-1}(B_H)$ has μ_H-a.s. continuous sample-paths. Consider

$$A_{\mathbf{Q}} = \bigcup_{t \in [0,1] \cap \mathbf{Q}} A_t.$$

It is a μ_H-negligeable set and for $\omega \in A_{\mathbf{Q}}^c$, for $t \in [0,1] \cap \mathbf{Q}$, $B(\omega, s) = \mathcal{K}_H^{-1}(B_H)(\omega, s)$. Thus, by continuity, this identity still holds for any $t \in [0,1]$ and any $\omega \in A_{\mathbf{Q}}^c$. This means that Eqn. (4.8) holds.

We now prove (4.9),

$$\begin{aligned}
\mathbf{E}\left[\psi \int_0^1 \mathcal{K}_H^{-1}(B_H) g(t)\ \mathrm{d}t\right] &= \int_0^1 \mathbf{E}\left[\psi\, B_H(t)\right] (\mathcal{K}_H^{-1})^*(g)(t)\ \mathrm{d}t \\
&= \int_0^1 \mathbf{E}\left[\psi\, \delta_{B_H}(R_H(t,.))\right] (\mathcal{K}_H^{-1})^*(g)(t)\ \mathrm{d}t \\
&= \mathbf{E}\left[\int_0^1 (\mathcal{K}_H^{-1})^*(g)(t) \int_0^1 \dot{\nabla}_s \psi\, K_H(t,s)\ \mathrm{d}s\ \mathrm{d}t\right] \\
&= \mathbf{E}\left[\int_0^1 \dot{\nabla}_s \psi \int_0^1 K_H(t,s)(\mathcal{K}_H^{-1})^*(g)(t)\ \mathrm{d}t\ \mathrm{d}s\right] \\
&= \mathbf{E}\left[\int_0^1 \dot{\nabla}_s \psi K_H^*(\mathcal{K}_H^{-1})^*(g))(s)\ \mathrm{d}s\right]
\end{aligned}$$

By the very definition of $\mathcal{K}_H$,

$$K_H^* \circ (\mathcal{K}_H^{-1})^* = K_H^* \circ (K_H^{-1})^* \circ K_{1/2}^* = K_{1/2}^*.$$

Thus, we have

$$\begin{aligned}
\mathbf{E}\left[\psi \int_0^1 \mathcal{K}_H^{-1}(B_H) g(t)\ \mathrm{d}t\right] &= \mathbf{E}\left[\int_0^1 \dot{\nabla}_s \psi\ K_{1/2}^* g(s)\ \mathrm{d}s\right] \\
&= \mathbf{E}\left[\int_0^1 \dot{\nabla}_s \psi \int_s^1 g(t)\ \mathrm{d}t\ \mathrm{d}s\right] \\
&= \mathbf{E}\left[\int_0^1 \int_0^1 \dot{\nabla}_s \psi\ g(t)\, \mathbf{1}_{[s,1]}(t)\ \mathrm{d}t\ \mathrm{d}s\right] \\
&= \mathbf{E}\left[\int_0^1 \int_0^1 \dot{\nabla}_s \psi\ g(t)\, \mathbf{1}_{[0,t]}(s)\ \mathrm{d}t\ \mathrm{d}s\right] \\
&= \mathbf{E}\left[\int_0^1 g(t) \int_0^1 \dot{\nabla}_s \psi\, \mathbf{1}_{[0,t]}(s)\ \mathrm{d}s\ \mathrm{d}t\right].
\end{aligned}$$

On the other hand, $B(t) = \delta_{B_H}\big(K_H(\mathbf{1}_{[0,t]})\big)$ hence,

$$\begin{aligned}\mathbf{E}\left[\psi \int_0^1 B(t)g(t)\ \mathrm{d}t\right] &= \mathbf{E}\left[\psi \int_0^1 \delta_{B_H}\big(K_H(\mathbf{1}_{[0,t]})\big)\, g(t)\ \mathrm{d}t\right] \\ &= \mathbf{E}\left[\int_0^1 g(t) \int_0^1 \dot{\nabla}_s \psi\ \mathbf{1}_{[0,t]}(s)\ \mathrm{d}s\ \mathrm{d}t\right].\end{aligned}$$

Then, (4.9) follows.

Since the operator involved in the previous relation are all lower triangular, we can go further and show that B and B_H generate the same filtration.

Definition 4.5. Recall that $(\dot{\pi}_t, t \in [0,1])$ are the projections defined by

$$\begin{aligned}\dot{\pi}_t\ :\ L^2([0,1] \to \mathbf{R};\, \lambda) &\longrightarrow L^2([0,1] \to \mathbf{R};\, \lambda) \\ f &\longmapsto f\mathbf{1}_{[0,t)}.\end{aligned}$$

Let V be a closable map from $\operatorname{Dom} V \subset L^2([0,1] \to \mathbf{R};\, \lambda)$ into $L^2([0,1] \to \mathbf{R};\, \lambda)$.

Then, V is $\dot{\pi}$-causal if $\operatorname{Dom} V$ is $\dot{\pi}$-stable, i.e. $\dot{\pi}_t \operatorname{Dom} V \subset \operatorname{Dom} V$ for any $t \in [0,1]$ and if for any $t \in [0,1]$,

$$\dot{\pi}_t V \dot{\pi}_t = \dot{\pi}_t V.$$

Consider also π_t^H defined by

$$\begin{aligned}\pi_t^H\ :\ \mathcal{H}_H &\longrightarrow \mathcal{H}_H \\ h &\longmapsto K_H\big(\pi_t K_H^{-1}(h)\big) = K_H\big(\dot{h}\,\mathbf{1}_{[0,t]}\big).\end{aligned}$$

Remark 4.4. An integral operator, i.e.

$$Vf(t) = \int_0^1 V(t,s)f(s)\ \mathrm{d}s$$

is $\dot{\pi}$-causal if and only if $V(t,s) = 0$ for $s > t$. For V_1, V_2 two causal operators, their composition V_1V_2 is still causal:

$$\begin{aligned}\pi_t V_1 V_2 \pi_t = (\pi_t V_1 \pi_t) V_2 \pi_t &= \pi_t V_1 (\pi_t V_2 \pi_t) \\ &= \pi_t V_1(\pi_t V_2) = (\pi_t V_1 \pi_t) V_2 = \pi_t V_1 V_2.\end{aligned}$$

Corollary 4.1. *The filtrations generated by B_H and B do coincide.*

Proof. From the representation

$$B_H(t) = \int_0^t K_H(t,s)\ \mathrm{d}B(s),$$

we deduce that

$$\sigma\{B_H(s),\ s\le t\}\subset\sigma\{B(s),\ s\le t\}.$$

We have $\mathcal{K}_H^{-1}=K_{1/2}K_H^{-1}$. From Theorem 4.3, K_H^{-1} appears as the composition of fractional derivatives and multiplication operators:

$$f\mapsto x^\alpha f.$$

Time derivatives of any order (as in Definition **??**) are local operators and as such are causal. It is straightforward that multiplication operators are also causal. Thus, $\mathcal{K}_H^{-1}$ appears as the composition of causal operators hence it is causal. This means that

$$B(t)=\int_0^t V(t,s)B_H(s)\ \mathrm{d}s$$

for some lower trianguler kernel V. Hence,

$$\sigma\{B_H(s),\ s\le t\}\supset\sigma\{B(s),\ s\le t\},$$

and the equality of filtrations is proved.

We can now reap the fruits of our not so usual presentation of the Malliavin calculus for the Brownian motion, in which we cautiously sidestepped chaos decomposition. Eqn. (4.5) is the exact analog of Eqn. (1.7) hence the Cameron-Martin Theorem can be proved identically:

Theorem 4.11. *For any $h\in\mathcal{H}_H$, for any bounded $F:\mathcal{W}\to\mathbf{R}$,*

$$\mathbf{E}\left[F(B_H+\mathfrak{e}(h)\right]=\mathbf{E}\left[F(B_H)\exp\left(\delta_{B_H}(h)-\frac{1}{2}\|h\|^2_{\mathcal{H}_H}\right)\right].\tag{4.10}$$

This entails the integration by parts formula, pending of (2.2): For any F and G in $\mathcal{S}_H$, for any $h\in\mathcal{H}_H$,

$$\mathbf{E}\left[G\left\langle\nabla F,h\right\rangle_{\mathcal{H}_H}\right]=-\mathbf{E}\left[F\left\langle\nabla G,h\right\rangle_{\mathcal{H}_H}\right]+\mathbf{E}\left[FG\,\delta_{B_H}h\right].\tag{4.11}$$

Definition 4.4 is formally the very same as Definition 2.1 so that the definition of the Sobolev spaces are identical.

Definition 4.6. The space $\mathbb{D}^H_{p,1}$ is the closure of $\mathcal{S}_H$ for the norm

$$\|F\|_{p,1,H}=\mathbf{E}\left[|F|^p\right]^{1/p}+\mathbf{E}\left[\|\nabla F\|^p_{\mathcal{H}_H}\right]^{1/p}.$$

The iterated gradient are defined likewise and so do the Sobolev of higher order, $\mathbb{D}_{p,k,H}$. We sill clearly have

$$\nabla(FG) = F\nabla G + G\nabla F$$
$$\nabla\phi(F) = \phi'(F)\nabla F$$

for $F \in \mathbb{D}_{p,1,H}$, $G \in \mathbb{D}_{q,1,H}$ and ϕ Lipschitz continuous. As long as we do not use the temporal scale, there is no difference between the identities established for the usual Brownian motion and that relative to the fractional Brownian motion.

Theorem 4.12. *For any F in $L^2(\mathcal{W} \to \mathbf{R};\, \mu_H)$,*

$$\Gamma(\pi_t^H)F = \mathbf{E}\left[F \,|\, \mathcal{F}_t^H\right],$$

in particular,

$$\mathbf{E}\left[W_t \,|\, \mathcal{F}_r^H\right] = \int_0^t K_H(t,s)\mathbf{1}_{[0,r]}(s)\,\delta B(s), \text{ and}$$
$$\mathbf{E}\left[\exp(\delta_{B_H}u - 1/2\|u\|^2_{\mathcal{H}_H}) \,|\, \mathcal{F}_t^H\right] = \exp(\delta_{B_H}\pi_t^H u - 1/2\|\pi_t^H u\|^2_{\mathcal{H}_H}),$$

for any $u \in \mathcal{H}_H$.

Proof. Let $\{h_n,\, n \ge 1\}$ be a denumerable family of elements of $\mathcal{H}_H$ and let $V_n = \sigma\{\delta_{B_H}h_k,\, 1 \le k \le n\}$. Denote by p_n the orthogonal projection on $\operatorname{span}\{h_1, \ldots, h_n\}$. For any f bounded, for any $u \in \mathcal{H}_H$, by the Cameron–Martin theorem we have

$$\begin{aligned}
\mathbf{E}\,[\Lambda_1^u\, f(\delta_{B_H}h_1, \ldots, \delta_{B_H}h_n)] \\
&= \mathbf{E}\,[f(\delta_{B_H}h_1(w+u), \ldots, \delta_{B_H}h_n(w+u))] \\
&= \mathbf{E}\,[f(\delta_{B_H}h_1 + (h_1,u)_{\mathcal{H}_H}, \ldots, \delta_{B_H}h_n + (h_n,u)_{\mathcal{H}_H})] \\
&= \mathbf{E}\,[f(\delta_{B_H}h_1(w+p_nu), \ldots, \delta_{B_H}h_n(w+p_nu))] \\
&= \mathbf{E}\,[\Lambda_1^{p_nu}\, f(\delta_{B_H}h_1, \ldots, \delta_{B_H}h_n)]\,,
\end{aligned}$$

hence

$$\mathbf{E}\,[\Lambda_1^u \,|\, V_n] = \Lambda_1^{p_nu}. \tag{4.12}$$

Choose h_n of the form $\pi_t^H(e_n)$ where $\{e_n,\, n \ge 1\}$ is an orthonormal basis of $\mathcal{H}_H$, i.e., $\{h_n,\, n \ge 1\}$ is an orthonormal basis of $\pi_t^H(\mathcal{H}_H)$. By the previous theorem, $\bigvee_n V_n = \mathcal{F}_t^H$ and it is clear that p_n tends pointwise to π_t^H, hence from (4.12) and martingale convergence theorem, we can conclude that

$$\mathbf{E}\left[\Lambda_1^u \,|\, \mathcal{F}_t^H\right] = \Lambda_1^{\pi_t^H u} = \Lambda_t^u.$$

Moreover, for $u \in \mathcal{H}_H$,

$$\Gamma(\pi_t^H)(\Lambda_1^u) = \Lambda_1^{\pi_t^H u},$$

hence by density of linear combinations of Wick exponentials, for any $F \in L^2(\mu_H)$,

$$\Gamma(\pi_t^H)F = \mathbf{E}\left[F \,|\, \mathcal{F}_t^H\right],$$

and the proof is completed.

Definition 4.7. For the sake of notations, we set, for $\dot{u}$ such that $K_H\dot{u}$ belongs to $\mathrm{Dom}_p\,\delta_{B_H}$ for some $p > 1$,

$$\int_0^1 \dot{u}(s)\delta B(s) = \delta_{B_H}(K_H\dot{u}) \text{ and } \int_0^t \dot{u}(s)\delta B(s) = \delta_{B_H}(\pi_t^H K_H\dot{u}). \quad (4.13)$$

Note that, for any $\psi \in \mathbb{D}_{p/(p-1),1}$

$$\mathbf{E}\left[\psi \int_0^1 \dot{u}(s)\delta B(s)\right] = \mathbf{E}\left[\int_0^1 \dot{\nabla}_s\psi\ \dot{u}(s)\ \mathrm{d}s\right].$$

The next result is the Clark formula. It reads formally as (3.11) but we should take care that the $\dot{\nabla}$ does not represent the same object. Here it is defined as $\dot{\nabla} = K_H^{-1}\nabla$.

Corollary 4.2. *For any $F \in L^2(\mathcal{W} \to \mathbf{R};\ \mu_H)$,*

$$F = \mathbf{E}\left[F\right] + \int_0^1 \mathbf{E}\left[\dot{\nabla}_s F \,|\, \mathcal{F}_s\right]\ \delta B(s).$$

Proof. With the notations at hand, Theorem 4.12 implies that

$$\begin{aligned}\mathbf{E}\left[\Lambda_1^h \,|\, \mathcal{F}_t\right] &= \exp\left(\delta_{B_H}(\pi_t^H h) - \frac{1}{2}\|\pi_t^H h\|_{\mathcal{H}_H}^2\right)\\ &= \exp\left(\int_0^t \dot{h}(s)\ \delta B(s) - \frac{1}{2}\int_0^t \dot{h}^2(s)\ \mathrm{d}s\right).\end{aligned}$$

This means that we have the usual relation

$$\Lambda_t^h = 1 + \int_0^t \Lambda_s\dot{h}(s)\ \delta B(s) = \mathbf{E}\left[\Lambda_1^h\right] + \int_0^1 \mathbf{E}\left[\dot{\nabla}_s\Lambda_1^h \,|\, \mathcal{F}_s\right]\ \delta B(s).$$

By density of the Doléans exponentials, we obtain the result.

Should we want to obfuscate everything, we could write

$$F = \mathbf{E}\left[F\right] + \delta_{B_H}\left(K_H\big(\mathbf{E}\left[(K_H^{-1}\nabla)_{.}F \,|\, \mathcal{F}_{.}\right]\big)\right).$$

4.2 Itô formula

Definition 4.8. Consider the operator $\mathcal{K}$ defined by $\mathcal{K} = I_{0^+}^{-1} \circ K_H$.

For $H > 1/2$, it is a continuous map from L^p into $I_{H-1/2,p}$, for any $p \geq 1$. Let $\mathcal{K}_t^*$ be its adjoint in $L^p([0,t])$, i.e. for any $f \in L^p([0,t])$, any g sufficently regular,

$$\int_0^t \mathcal{K}f(s)\, g(s)\ \mathrm{d}s = \int_0^t f(s)\, \mathcal{K}_t^* g(s)\ \mathrm{d}s.$$

The map $\mathcal{K}_t^*$ is continuous from $\big(I_{0^*}^{H-1/2}(L^p([0,t]))\big)^*$ into $L^p([0,t])$.

Theorem 4.13. *Assume $H > 1/2$. For $f \in \mathcal{C}_b^2$,*

$$f(B_H(t)) = f(0) + \int_0^t \mathcal{K}_t^*\big(f' \circ B_H\big)(s)\ \delta B(s) + H\, V_H \int_0^1 f''\big(B_H(s)\big) s^{2H-1}\ ds.$$

Proof. Introduce the function g as

$$g(x) = f(\frac{a+b}{2} + x) - f(\frac{a+b}{2} - x).$$

This function is even, satisfies

$$g^{(2j+1)}(0) = 2f^{(2j+1)}((a+b)/2) \text{ and } g(\frac{b-a}{2}) = f(b) - f(a).$$

Apply the Taylor formula to g between the points 0 and $(b-a)/2$ to get

$$\begin{aligned} f(b) - f(a) = \sum_{j=0}^{n} \frac{2^{-2j}}{(2j+1)!}(b-a)^{2j+1} f^{(2j+1)}(\frac{a+b}{2}) \\ + \frac{(b-a)^{2(n+1)}}{2} \int_0^1 \lambda^{2n+1} g^{(2(n+1))}(\lambda a + (1-\lambda)b)\mathrm{d}\lambda. \end{aligned}$$

For any $\psi \in \mathcal{E}$ of the form $\psi = \exp(\delta_{B_H} h - \frac{1}{2}\|h\|^2_{\mathcal{H}_H})$ with $h \in \mathcal{C}_b^1 \subset \mathcal{H}_H$. Note that ψ satisfies $\nabla \psi = \psi\, h \in L^2(W; \mathcal{C}_b^1)$. Since $\mathcal{C}_b^1$ is dense into $\mathcal{H}_H$, these functionals are dense in $L^2(\mathcal{W})$. We thus have

$$\begin{aligned} &\mathbf{E}\left[\big(f\big(B_H(t+\varepsilon)\big) - f\big(B_H(t)\big)\big)\, \psi\right] \\ &= \mathbf{E}\left[\big(B_H(t+\varepsilon) - B_H(t)\big)\ f'\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\ \psi\right] \\ &+ \frac{1}{2}\mathbf{E}\left[\big(B_H(t+\varepsilon) - B_H(t)\big)^2 \int_0^1 r\, g^{(2)}(rB_H(t) + (1-r)B_H(t+\varepsilon))\mathrm{d}r\ \psi\right] \\ &= A_0 + \frac{1}{2}A_1. \quad (4.14) \end{aligned}$$

For A_0, we have

$$\begin{aligned}A_0 &= \mathbf{E}\left[\big(B_H(t+\varepsilon)-B_H(t)\big)\, f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\ \psi\right]\\ &= \mathbf{E}\left[\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\delta B(s)\ f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\ \psi\right]\\ &= \mathbf{E}\left[\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\ \dot\nabla_s\left(f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\ \psi\right)\ \mathrm{d}s\right].\end{aligned}$$

Since $\dot\nabla$ is a true derivation operator

$$\begin{aligned}\dot\nabla_s\left(f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\ \psi\right) &= f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\dot\nabla_s\psi\\ &\quad + f''\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\big(K_H(t+\varepsilon,s)+K_H(t,s)\big).\end{aligned}$$

Thus,

$$\begin{aligned}A_0 &= \mathbf{E}\left[f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\dot\nabla_s\psi\ \mathrm{d}s\right]\\ &+ \mathbf{E}\left[\psi\, f''\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\right.\\ &\qquad \left.\times\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\big(K_H(t+\varepsilon,s)+K_H(t,s)\big)\ \mathrm{d}s\right]\\ &= B_1+B_2.\end{aligned}$$

By the very definition of $\dot\nabla$,

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\dot\nabla_s\psi\ \mathrm{d}s &= \frac{1}{\varepsilon}\big(\nabla\psi(t+\varepsilon)-\nabla\psi(t)\big)\\ &\xrightarrow{\varepsilon\to 0} \frac{d}{dt}\nabla\psi(t) = I_{0^+}^{-1}\circ K_H(\dot\nabla\psi)(t) = \mathcal{K}(\dot\nabla\psi)(t).\end{aligned}$$

Moreover, since $\nabla\psi$ belongs to $L^2(W; I_{H+1/2,2})$,

$$\mathbf{E}\left[|\nabla\psi(t+\varepsilon)-\nabla\psi(t)|^2\right] \le c\,\|\mathcal{K}\dot\nabla\psi\|_{L^2(W;I_{H-1/2,2})}\ |\varepsilon|.$$

Hence,

$$\varepsilon^{-1}\, B_1 \xrightarrow{\varepsilon\to 0} \mathbf{E}\left[f'(B_H(t))\,\mathcal{K}\dot\nabla\psi(t)\right].$$

Simple calculations give that

$$B_2 = \mathbf{E}\left[\psi\, f''\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\ \Big(R_H(t+\varepsilon,t+\varepsilon)-R_H(t,t)\Big)\right]$$

and that

$$\varepsilon^{-1}\Big(R_H(t+\varepsilon,t+\varepsilon)-R_H(t,t)\Big)=V_H\,\frac{(t+\varepsilon)^{2H}-t^{2H}}{\varepsilon}\xrightarrow{\varepsilon\to 0}2H\,V_H\,t^{2H-1}.$$

The dominated convergence theorem then yields

$$\varepsilon^{-1}B_2\xrightarrow{\varepsilon\longrightarrow 0}H\,V_H\mathbf{E}\left[\psi f''(B_H(t))\,t^{2H-1}\right].$$

If $H>1/2$, $\varepsilon^{-1}A_1$ does vanish. Actually, recall that $B_H(t+\varepsilon)-B_H(t)$ is a centered Gaussian random variable of variance proportional to ε^{2H}, hence

$$\varepsilon^{-1}|A_1|\le c\mathbf{E}\left[|B_H(t+\varepsilon)-B_H(t)|^2\right]\|f^{(2)}\|_{L^\infty}\le c\,\varepsilon^{2H-1}\,\|f^{(2)}\|_{L^\infty}\xrightarrow{\varepsilon\to 0}0,$$

since $2H-1>0$.

We have proved so far that

$$\frac{d}{dt}\mathbf{E}\left[\psi\,f'\big(B_H(t)\big)\right]=\mathbf{E}\left[f\big(B_H(t)\big)\,\nabla\psi(t)\right]+H\,V_H\mathbf{E}\left[\psi\;f''\big(B_H(t)\big)\,t^{2H-1}\right].\tag{4.15}$$

It is straightforward that the right-hand-side of (4.15) is continuous as a function of t on any interval $[0,T]$. Hence we can integrate the previous relation and we get

$$\begin{aligned}\mathbf{E}\left[\psi\,f\big(B_H(t)\big)\right]-\mathbf{E}\left[\psi\,f\big(B_H(0)\big)\right]&=\mathbf{E}\left[\int_0^t f'\big(B_H(s)\big)\,\mathcal{K}\dot\nabla\psi(s)\,\mathrm{d}s\right]\\&\quad+H\,V_H\;\mathbf{E}\left[\psi\int_0^t\;f''\big(B_H(s)\big)\,s^{2H-1}\,\mathrm{d}s\right].\end{aligned}$$

Remark now that

$$\begin{aligned}\mathbf{E}\left[\int_0^t f'\big(B_H(s)\big)\,\mathcal{K}\dot\nabla\psi(s)\,\mathrm{d}s\right]&=\mathbf{E}\left[\int_0^1 f'\big(B_H(s)\big)\mathbf{1}_{[0,t]}(s)\,\mathcal{K}\dot\nabla\psi(s)\,\mathrm{d}s\right]\\=\mathbf{E}\left[\int_0^1\mathcal{K}_1^*\big(f'\circ B_H\,\mathbf{1}_{[0,t]}\big)\,\dot\nabla_s\psi\,\mathrm{d}s\right]&=\mathbf{E}\left[\psi\int_0^1\mathcal{K}_1^*\big(f'\circ B_H\,\mathbf{1}_{[0,t]}\big)(s)\;\delta B(s)\right].\end{aligned}$$

Note that

$$\mathcal{K}_1^*(f'\mathbf{1}_{[0,t]})(s)=\frac{d}{ds}\int_s^1 K(r,s)f'(r)\mathbf{1}_{[0,t]}(r)\,\mathrm{d}r=0\text{ if }s>t.$$

This means that

$$\pi_t^H\left(\mathcal{K}_t^*(f'\mathbf{1}_{[0,t]})\right)=\mathcal{K}_t^*(f'\mathbf{1}_{[0,t]})$$

and by the definition (4.13),